Weak mixing and mixing of a single transformation of a topological (semi)group

T.K. Subrahmonian Moothathu

Department of Mathematics and Statistics, University of Hyderabad, Hyderabad 500 046, India.

Email: tksm@uohyd.ernet.in

May 31, 2009

Abstract

We investigate some aspects of the iterative dynamics of a single continuous homomorphism $T : X \to X$ of a Hausdorff topological (semi)group X. We show that if X is a Hausdorff topological group and $T : X \to X$ is a continuous homomorphism such that either T is syndetically transitive, or T is non-wandering with a dense set of points having orbits converging to the identity element, then T is topologically weak mixing. We also show that for some familiar topological (semi)groups X, there is an (invertible) element $a \in X$ such that $T : X \to X$ given by $T(x) = axa^{-1}$ is topologically mixing. As a corollary we get a zero-one law for generic dynamics on certain spaces such as the Cantor space, the Hilbert cube and $\mathbb{R}^k$.

Key words: Topological transitivity, weak mixing, mixing, syndetic set, topological group, topological semi-group, generic dynamics.
MSC 2000: 54H20.

1 Introduction

In topological dynamics, we have on one hand the theory of group actions on topological groups and on the other hand the theory of a single continuous transformation of a Hausdorff topological space. The former possesses many powerful tools where as the latter is more delicate and often forces one to work from the scratch. In this note we consider a class of dynamical systems which is somewhere in between, and thereby hope that without sacrificing the delicacy of the system too much we will have enough tools to work with. Namely, we consider the dynamics of a single continuous homomorphism $T : X \to X$ of a Hausdorff topological (semi)group X.

Mainly we concentrate on stronger forms of transitivity such as weak mixing and mixing. After the necessary preliminaries, in section 3 we show that for a continuous homomorphism T of a topological group, if either T is syndetically transitive, or T is non-wandering with a dense set of points having orbits converging to the identity element, then T is topologically weak mixing. In section 4, we investigate the dynamics of the conjugacy map $x \mapsto axa^{-1}$ for a fixed element a and show that in many familiar situations this map is topologically mixing. As a corollary we get a zero-one law for *generic dynamics* on certain spaces such as the Cantor space, the Hilbert cube and $\mathbb{R}^k$.

2 Preliminaries

For a subset $C \subset \mathbb{N}$ we recall two notions of "largeness". The set C is said to be **thick** if it contains arbitrarily large blocks of consecutive integers, and C is said to be **syndetic** if it is an infinite set with bounded gaps, i.e., $C = \{c_1 < c_2 < \cdots\}$ is infinite with $\sup_{k\in\mathbb{N}}(c_{k+1} - c_k) < \infty$. Note that $C \subset \mathbb{N}$ is syndetic iff C intersects every thick subset of $\mathbb{N}$.

Let $T : X \to X$ be a continuous map of a Hausdorff topological space X. The T-**orbit** $\{x, T(x), T^2(x), \ldots\}$ of a point $x \in X$ will be denoted by $O_T(x)$. For $x \in X$ and $U, V \subset X$, let

$$N_T(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}, \tag{1}$$

$$N_T(U, V) = \{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}. \tag{2}$$

The following fact can easily be proved using the Baire Category Theorem:

Proposition 1. *Let X be a complete, separable metric space without isolated points. Then for a continuous map $T : X \to X$ the following are equivalent:*
(i) T is transitive, i.e., $N_T(U, V) \neq \emptyset$ for any two nonempty open sets $U, V \subset X$.
(ii) $\overline{O_T(X)} = X$ for some $x \in X$.
(iii) $\{x \in X : \overline{O_T(X)} = X\}$ is a dense G_δ subset of X.

We say $T : X \to X$ is **weak mixing** if $T \times T : X^2 \to X^2$ is transitive (where $X^2 = X \times X$). The following Proposition collects a few well-known facts about weak mixing, see also [5, 2].

Proposition 2. *Let $T : X \to X$ be a continuous map of a Hausdorff topological space X. The following are equivalent:*
(i) T is weak mixing.
(ii) $N_T(U, V)$ is thick for any two nonempty open sets $U, V \subset X$.
(iii) $N_T(U, V) \cap N_T(V, V) \neq \emptyset$ for any two nonempty open sets $U, V \subset X$.

A continuous map $T : X \to X$ is said to be **syndetically transitive** (respectively **mixing**) if $N_T(U, V)$ is syndetic (respectively cofinite) for any two nonempty open sets $U, V \subset X$. See [11, 12] for some interesting properties of syndetically transitive maps.

As usual, by a **topological (semi)group** we mean a Hausdorff topological space with a (semi)group structure where the (semi)group operations are continuous. While discussing transitivity of continuous homomorphisms of topological (semi)groups, sometimes we would like to put the extra assumption that the topological (semi)group admits a complete separable metric, in view of Proposition 1. If this extra assumption holds, the topological (semi)group will be referred to as a **Polish (semi)group**.

Convention: *To avoid pathologies all topological groups/semigroups under consideration are assumed to be infinite and without any isolated points.*

3 Syndetical transitivity implies weak mixing

This section contains two main results of similar nature, the first of which is stated below.

Theorem 1. *Let X be a Hausdorff topological group and $T : X \to X$ be a continuous homomorphism. If T is syndetically transitive, then T is weak mixing.*

In order to appreciate this result, we make a few remarks. First of all, syndetical transitivity need not imply weak mixing for continuous maps of compact metric spaces, as is witnessed by any *irrational rotation* of the unit circle. Secondly, there are at least two natural classes of syndetically transitivity maps: (i) any transitive map admitting an invariant Borel probability measure of full support is syndetically transitive (c.f. [7]), (ii) any transitive map possessing a dense set of **almost periodic points** is clearly syndetically transitive, where $x \in X$ is called almost periodic if $N_T(x, U)$ is syndetic for any neighborhood U of x; and in particular any transitive map with a dense set of periodic points is syndetically transitive.

We also remark that syndetical transitivity is strictly stronger than weak mixing for continuous homomorphisms of Hausdorff topological groups, or even separable Hilbert spaces. To justify this assertion, we argue as follows. In [13], there is an example of a pair of continuous linear operators $T, S : X \to X$ on a separable Hilbert space X such that T, S are transitive but $T \times S$ is not transitive. Later it was observed that T, S are in fact weak mixing (c.f. [4]). Now, we claim that neither T nor S can be syndetically transitive. This is because the product of a syndetically transitive map with any weak mixing map should be transitive (use the characterization of weak mixing in terms of thick sets stated in Proposition 2).

To prove Theorem 1, we go for two Lemmas.

Lemma 1. *Let X be a Hausdorff topological group with identity element e, and $T : X \to X$ be a continuous homomorphism. Write $N(U, V) = N_T(U, V)$. Then the following are equivalent:*
(i) T is weak mixing.
(ii) For any three nonempty open sets $U, V, W \subset X$ with $e \in W$, $N(U, W) \cap N(V, V) \neq \emptyset$.

Proof. Only the implication (ii)$\Rightarrow$ (i) needs a proof. Let $U, V \subset X$ be nonempty open sets. It suffices to show that $N(U, V) \cap N(V, V) \neq \emptyset$ because of Proposition 2. Using the basic properties of a topological group, choose nonempty open sets $W, V_0, Y \subset X$ such that $e \in W$, $V_0 \subset V$, $V_0 W \subset V$ and $V_0 Y \subset U$. By (ii), there is some $n \in N(Y, W) \cap N(V_0, V_0)$. Consider points $y \in Y$ and $v \in V_0$ with $T^n(y) \in W$ and $T^n(v) \in V_0$. Then, $vy \in V_0 Y \subset U$ and $T^n(vy) = T^n(v)T^n(y) \in V_0 W \subset V$ so that $n \in N(U, V) \cap N(V, V)$. □

Lemma 2. *Let X be a Hausdorff topological group with identity element e, and $T : X \to X$ be a continuous homomorphism. If T is transitive, then for any nonempty open set $U \subset X$ and any neighborhood W of e, $N_T(U, W)$ is thick.*

Proof. Given $k \in \mathbb{N}$, let V be the neighborhood of e defined as $V = \bigcap_{i=0}^{k} T^{-i}(W)$. Then $N_T(U, V) \neq \emptyset$ by transitivity and $N_T(U, V) + i \subset N_T(U, W)$ for $0 \leq i \leq k$. □

Proof of Theorem 1. Let $U, V, W \subset X$ be nonempty open sets and assume that the identity element e belongs to W. Then $N_T(U, W)$ is thick by Lemma 2, and $N_T(V, V)$ is syndetic by syndetical transitivity. Hence $N_T(U, W) \cap N_T(V, V) \neq \emptyset$ and therefore T is weak mixing by Lemma 1. □

Corollary 1. *Let $T : X \to X$ be a continuous homomorphism of a compact Hausdorff topological group X. If T is transitive, then T is weak mixing.*

Proof. Since T is transitive and X is compact, T is surjective. Now, a surjective continuous homomorphism of a compact Hausdorff topological group preserves the normalized Haar measure (c.f. [14]). By a remark above, T is syndetically transitive, so that Theorem 1 applies. □

To state the second main result of this section, we need more terminology. Let X be a Hausdorff topological group and $T : X \to X$ be a continuous homomorphism. The identity element of X will be denoted by e. We say $(x, y) \in X^2$ is a **(left-)proximal pair** for T if $e \in \overline{O_T(x^{-1}y)}$; and $(x, y) \in X^2$ is called a **(left-)asymptotic pair** for T if $\lim_{n\to\infty} T^n(x^{-1}y) = e$. Observe that if X is metrizable and if there is a left-invariant metric d on X, then (x, y) is a proximal pair iff $\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0$ and (x, y) is an asymptotic pair iff $\lim_{n\to\infty} d(T^n(x), T^n(y)) = 0$. For a given T, the relation of being an asymptotic pair is an equivalence relation on X, whereas the relation of being a proximal pair need not be, even though it is reflexive and symmetric. For $x \in X$, the **asymptotic cell** and the **proximal cell** of x are defined as

$$Asym(T, x) = \{y \in X : (x, y) \text{ is an asymptotic pair for } T\}, \tag{3}$$

$$Prox(T, x) = \{y \in X : (x, y) \text{ is a proximal pair for } T\}. \tag{4}$$

The following elementary properties are easy to derive; for example, to prove the statement (iii) below, use the continuity of T at $e = T(e)$.

Proposition 3. *Let X be a Hausdorff topological group and $T : X \to X$ be a continuous homomorphism. Then for every $x \in X$,*
(i) $Asym(T, x) = x \cdot Asym(T, e)$ and $Prox(T, x) = x \cdot Prox(T, e)$.
(ii) $Asym(T, e)$ is a subgroup of X.
(iii) If $x \in Prox(T, e)$, then $N_T(x, W)$ is thick for any neighborhood W of e.
(iv) $Prox(T, x)$ is a G_δ subset of X when X is metrizable.

For a continuous map T of a compact metric space (X, d), it is a non-trivial result [1] that if T is weak mixing, then the proximal cell $Prox(T, x) := \{y \in X : \liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0\}$ is a dense G_δ in X

for every $x \in X$. Here 'weak mixing' cannot be replaced by 'transitive' since there exists a transitive map T of a compact metric space X (e.g. an *irrational rotation* of the unit circle) such that $Prox(T, x) = \{x\}$ for every $x \in X$. However, as shown below, for a continuous homomorphism of a Polish group, we get that $Prox(T, x)$ is "big" for every x just from the assumption of transitivity.

Proposition 4. *Let X be a Polish group and $T : X \to X$ be a continuous homomorphism. If T is transitive, then for any $k \in \mathbb{N}$, every element of X^k has a dense G_δ proximal cell in X^k under the action of $\underbrace{T \times \cdots \times T}_{k-times}$.*

Proof. Let $e(k)$ denote the identity element of X^k. In the light of Proposition 3, it suffices to show that the proximal cell of $e(k)$ is dense in X^k. Since X is Polish, $\overline{O_T(x)} = X$ for some $x \in X$, by Proposition 1. Then $[O_T(x)]^k$ is dense in X^k. Moreover, $[O_T(x)]^k$ is contained in the proximal cell of $e(k)$ by Proposition 3(iii). □

A continuous map $T : X \to X$ of a Hausdorff topological space is said to be **non-wandering** if $N_T(U, U) \neq \emptyset$ for any nonempty open set $U \subset X$. Clearly any transitive map is non-wandering. If X has no isolated points, and if $T : X \to X$ is non-wandering, then $N_T(U, U)$ can easily be seen to be infinite for any nonempty open set $U \subset X$. We may say T is **syndetically non-wandering** if $N_T(U, U)$ is syndetic for any nonempty open set $U \subset X$. The second main result of the section is the following.

Theorem 2. *Let X be a Hausdorff topological group with identity element e, and $T : X \to X$ be a continuous homomorphism. Suppose that either T is non-wandering with $\overline{Asym(T, e)} = X$, or T is syndetically non-wandering with $\overline{Prox(T, e)} = X$. Then T is weak mixing.*

Proof. Let W be any neighborhood of e. First note that if $x \in Asym(T, e)$ then $N_T(x, W)$ is cofinite and if $x \in Prox(T, e)$ then $N_T(x, W)$ is thick. Apply Lemma 1. □

Using Theorem 2 we can improve a recent result of Grosse-Erdmann and Peris [10]. For $C \subset \mathbb{N}$, the **upper density** $\overline{\rho}(C)$ and the **upper Banach density** $\rho^*(C)$ are defined as

$$\overline{\rho}(C) := \limsup_{n\to\infty} \frac{|C \cap \{1, \ldots, n\}|}{n}, \tag{5}$$

$$\rho^*(C) := \limsup_{0<n-k\to\infty} \frac{|C \cap \{k, k+1, \ldots, n-1\}|}{n-k}. \tag{6}$$

Note that $\rho^*(C) \geq \overline{\rho}(C)$. In [10], the authors essentially proved that if $T : X \to X$ is a continuous linear operator of an $\mathcal{F}$-space (i.e., a complete separable topological vector space) X and if there is $x \in X$ such that $\overline{\rho}(N_T(x, U)) > 0$ for every nonempty open set $U \subset X$, then T is weak mixing. Now, it is a fact that if $\rho^*(C) > 0$, then the difference set $C - C := \{n - m > 0 : n, m \in C\}$ is syndetic; see [3] for a stronger statement. It follows that $N_T(U, U)$ is syndetic if $\rho^*(N_T(x, U)) > 0$ ($\because$ if $m < n$ and if $T^m(x), T^n(x) \in U$, then $T^{n-m}(U) \cap U \neq \emptyset$). Moreover, if $\rho^*(N_T(x, U)) > 0$ for every nonempty open set $U \subset X$, then $O_T(x)$ is dense and hence $Prox(T, e)$ is also dense, by Proposition 3(iii). Thus we have the following improvement to the result in [10]:

Corollary 2. *Let X be a Hausdorff topological group and $T : X \to X$ be a continuous homomorphism. If there is $x \in X$ such that $\rho^*(N_T(x, U)) > 0$ for every nonempty open set $U \subset X$, then T is weak mixing.*

Note that the existence of an element $x \in X$ as in the hypothesis of Corollary 2 is guaranteed if X is Polish and T has an ergodic Borel probability measure of full support, by Birkhoff's Theorem (c.f. [14]).

4 Mixing of the conjugacy map $x \mapsto axa^{-1}$

In this section we prove that for some familiar topological (semi)groups X, there exists an (invertible) element $a \in X$ such that the map $T : X \to X$ given by $T(x) = axa^{-1}$ is mixing. First we present a sufficient condition for mixing on Hausdorff topological semigroups, which is adapted from from the theory of *hypercyclic operators*, see [4].

Lemma 3. *Let X be a Hausdorff topological semigroup possessing an identity element e, and let $T : X \to X$ be a continuous homomorphism. Suppose that there exist dense subsets $A, B \subset X$ and a sequence of maps $S_n : B \to X$ (need not even be continuous) such that $\lim_{n\to\infty} T^n(a) = e = \lim_{n\to\infty} S_n(b)$ and $\lim_{n\to\infty} T^n(S_n(b)) = b$ for every $a \in A$ and every $b \in B$. Then T is mixing.*

Proof. The proof is essentially the same as the one usually presented in the $\mathcal{F}$-space setting in connection with the *Hypercyclicity Criterion* (see [4]). Our contribution is only the observation that the same proof goes through even for Hausdorff topological semigroups. Let $U, V \subset X$ be nonempty open sets. Let $a \in A \cap U$ and $b \in B \cap V$. Let $c_n = a \cdot S_n(b)$. Since $S_n(b) \to e$, we have $\lim_{n\to\infty} c_n = a$ and hence there exists $n_1 \in \mathbb{N}$ such that $c_n \in U$ for all $n \geq n_1$. Similarly, since $T^n(a) \to e$ and $T^n(S_n(b)) \to b$, we have $T^n(c_n) = T^n(a) \cdot T^n(S_n(b)) \to b$ as $n \to \infty$, and hence there exists $n_2 \in \mathbb{N}$ such that $T^n(c_n) \in V$ for all $n \geq n_2$. Thus $n \in N_T(U, V)$ for all $n \geq \max\{n_1, n_2\}$. $\square$

If X is a compact metric space or even a locally compact second countable space, let $H(X) = H(X, X)$ and $C(X) = C(X, X)$ respectively be the set of all self-homeomorphisms of X and the set of all continuous self-maps of X. With respect to the compact-open topology and the binary operation of composition of maps, $H(X)$ is a Polish group. Similarly $C(X)$ is a Polish semigroup possessing an identity element, namely the identity map of X. Now, $H(X)$ acts on itself and on $C(X)$ by conjugation. A Borel subset $\mathcal{P} \subset C(X)$ is said to be a **dynamical property** on X if $\mathcal{P}$ is invariant under this conjugacy action, i.e., if $h\mathcal{P}h^{-1} \subset \mathcal{P}$ for every $h \in H(X)$. The following are some examples of dynamical properties: $\mathcal{P}_t := \{f \in C(X) : f \text{ is transitive}\}$, $\mathcal{P}_w := \{f \in C(X) : f \text{ is weak mixing}\}$, and $\mathcal{P}_m := \{f \in C(X) : f \text{ is mixing}\}$. (Clearly they are invariant under the conjugacy action; moreover $\mathcal{P}_t$, $\mathcal{P}_w$ are G_δ and $\mathcal{P}_m$ is $F_{\sigma\delta}$ in $C(X)$.)

In [8], Glasner and Weiss proved that if X is either the Cantor space or the Hilbert cube, then the conjugacy action of $H(X)$ on itself has a dense orbit, i.e., there exists $g \in H(X)$ such that $\{hgh^{-1} : h \in H(X)\}$ is dense in $H(X)$. Combined with a *zero-one law* from [6], this implies that for any dynamical property $\mathcal{P}$ on X, either $H(X) \cap \mathcal{P}$ or $H(X) \setminus \mathcal{P}$ is residual in $H(X)$. We prove below a similar result for

$C(X)$ in a very strong form. We show that there is $\sigma \in H(X)$ such that the map $f \mapsto \sigma f \sigma^{-1}$ on $C(X)$ is mixing.

Note that the Cantor space can be realized as $\{0,1\}^{\mathbb{Z}}$ and the Hilbert cube can be realized as $[0,1]^{\mathbb{Z}}$.

Theorem 3. *Let Y be a compact metric space with at least two points, let $X = Y^{\mathbb{Z}}$ and let $\sigma : X \to X$ be the shift homeomorphism. Then the continuous homomorphism $T : C(X) \to C(X)$ given by $T(f) = \sigma f \sigma^{-1}$ is mixing.*

Proof. For $i \in \mathbb{Z}$, let $\pi_i : X \to Y$ be the projection map to the i^{th} coordinate. Note that any $f \in C(X)$ is of the form $f = (f_i)_{i\in\mathbb{Z}}$ where $f_i = \pi_i f : X \to Y$. If d is an admissible metric on Y, then an admissible metric $\tilde{d}$ on $C(X)$ is given by

$$\tilde{d}(f,g) = \sup\{\sum_{i\in\mathbb{Z}} \frac{d(f_i(x), g_i(x))}{2^{|i|}} : x \in X\}.$$

For $k \in \mathbb{N}$, let $A_k = \{f \in C(X) : f_i = \pi_i \text{ for all } i \in \mathbb{Z} \text{ with } |i| \geq k\}$ and let $A = \bigcup_{k=1}^{\infty} A_k$. Now, given $f \in C(X)$ we can define $g = (g_i)_{i\in\mathbb{Z}} \in A_k$ by the requirement that $g_i = f_i$ for $|i| < k$ and $g_i = \pi_i$ for $|i| \geq k$, and it is clear that g is sufficiently close to f if k is large enough. Thus A is dense in $C(X)$.

The map T is invertible and $[T^n(f)]_i = f_{i+n}\sigma^{-n}$ for $f \in C(X)$. Consequently, for every $f \in A$ both $(T^n(f))$ and $(T^{-n}(f))$ converge in $C(X)$ to the identity map of X as $n \to \infty$. Therefore, by taking $B = A$ and $S_n = T^{-n}$ in Lemma 3, we deduce that T is mixing. ☐

In particular, by the *zero-one law* from [6], we obtain:

Corollary 3. *Let Y be a compact metric space with at least two points, let $X = Y^{\mathbb{Z}}$ and let $\mathcal{P}$ be a dynamical property on X. Then, either $\mathcal{P}$ or $C(X) \setminus \mathcal{P}$ is residual in $C(X)$.*

With the experience of Theorem 3, we may ask whether there are more natural examples of topological (semi)groups X possessing an (invertible) element $a \in X$ such that the continuous homomorphism $T : X \to X$ given by $T(x) = axa^{-1}$ is mixing. Below we indicate three such examples. The basic way in which Lemma 3 is invoked is the same.

Example 1. Let $\mathbb{S}(\mathbb{Z})$ be the Polish group of all permutations of $\mathbb{Z}$; see for instance [9] to find some recent developments concerning $\mathbb{S}(\mathbb{Z})$. Let $\sigma \in \mathbb{S}(\mathbb{Z})$ be the shift permutation. Define $T : \mathbb{S}(\mathbb{Z}) \to \mathbb{S}(\mathbb{Z})$ by $T(f) = \sigma f \sigma^{-1}$. The set $A \subset \mathbb{S}(\mathbb{Z})$ of all finitely supported permutations is dense in $\mathbb{S}(\mathbb{Z})$, and for any $f \in A$ both $(T^n(f))$ and $(T^{-n}(f))$ converge in $\mathbb{S}(\mathbb{Z})$ to the identity permutation as $n \to \infty$. Hence T is mixing.

Example 2. Let $C(\mathbb{R}^k)$ be the Polish semigroup of all continuous self-maps of $\mathbb{R}^k$. For any fixed non-zero $a \in \mathbb{R}^k$, let $\sigma \in C(\mathbb{R}^k)$ be the translation given by $\sigma(x) = x + a$. Define $T : C(\mathbb{R}^k) \to C(\mathbb{R}^k)$ by $T(f) = \sigma f \sigma^{-1}$. The set $A \subset C(\mathbb{R}^k)$ of all $f \in C(\mathbb{R}^k)$ such that $f(x) = x$ outside some bounded subset of $\mathbb{R}^k$, is dense in $C(\mathbb{R}^k)$, and for any $f \in A$ both $(T^n(f))$ and $(T^{-n}(f))$ converge in $C(\mathbb{R}^k)$ to the identity map of $\mathbb{R}^k$ as $n \to \infty$. Hence T is mixing. Here also we can apply the *zero-one law* from [6] and conclude that for any dynamical property $\mathcal{P}$ on $\mathbb{R}^k$, either $\mathcal{P}$ or $C(\mathbb{R}^k) \setminus \mathcal{P}$ is residual in $C(\mathbb{R}^k)$.

Example 3. This is similar to the previous example. Let $H_+(\mathbb{R})$ be the Polish group of all increasing homeomorphisms of $\mathbb{R}$. Let $T : H_+(\mathbb{R}) \to H_+(\mathbb{R})$ be $T(f) = \sigma f \sigma^{-1}$ where $\sigma \in H_+(\mathbb{R})$ is any non-zero translation. To deduce T is mixing, use the fact the set of all compactly supported homeomorphisms of $\mathbb{R}$ is dense in $H_+(\mathbb{R})$.

References

[1] E.Akin and S.Kolyada, *Li-Yorke sensitivity*, Nonlinearity, 16, (2003), 1421-1433.

[2] J.Banks, *Topological mapping properties defined by digraphs*, Discrete & Cont. Dyn. Sys., 5, (1999), 83-92.

[3] V.Bergelson, B.Host and B.Kra, *Multiple recurrence and nilsequences*, Invent. math., 160, (2005), 261-303.

[4] J.Bès and A.Peris, *Hereditarily Hypercyclic Operators*, J. Funct. Anal., 167, (1999), 94-112.

[5] H.Furstenberg, *Disjointness in ergodic theory, minimal sets and a problem in Diophantine approximation*, Math. Sys. Th., 1, (1967), 1-49.

[6] E.Glasner and J.King, *A zero-one law for dynamical properties*, Topological Dynamics and Applications (A volume in honor of Robert Ellis), Contemp. Math., 215, Amer. Math. Soc., Providence, RI, (1998), 215242.

[7] E.Glasner and B.Weiss, *Locally equicontinuous dynamical systems*, Colloq. Math., 84-85, (2000), 345-361.

[8] E.Glasner and B.Weiss, *The topological Rohlin property and topological entropy*, Amer. J. Math., 123(6), (2001), 1055-1070.

[9] E.Glasner and B.Weiss, *Minimal actions of the group* $\mathbb{S}(\mathbb{Z})$ *of permutations of the integers*, Geom. Funct. Anal., 12(5), (2002), 964-988.

[10] K.-G.Grosse-Erdmann and A.Peris, *Frequently dense orbits*, C. R. Acad. Sci. Paris, Ser. I, 341, (2005), 123-128.

[11] W.Huang and X.Ye, *Topological complexity, return times and weak disjointness*, Ergod. Th. & Dyn. Sys., 24, (2004) 825-846.

[12] T.K.S.Moothathu, *Stronger forms of sensitivity for dynamical systems*, Nonlinearity, 20, (2007), 2115-2126.

[13] H.Salas, *A hypercyclic operator whose adjoint is also hypercyclic*, Proc. Amer. Math. Soc., 112(3), (1991), 765-770.

[14] P.Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Math., Springer, (1982).