Stronger forms of sensitivity for dynamical systems

T.K. Subrahmonian Moothathu

tksm@uohyd.ernet.in

Abstract

For continuous self-maps of compact metric spaces, we initiate a preliminary study of stronger forms of sensitivity formulated in terms of 'large' subsets of $\mathbb{N}$. Mainly we consider 'syndetic sensitivity' and 'cofinite sensitivity'. We establish the following: (i) any syndetically transitive, non-minimal map is syndetically sensitive (this improves the result that sensitivity is redundant in Devaney's definition of chaos), (ii) any sensitive map of $[0, 1]$ is cofinitely sensitive, (iii) any sensitive subshift of finite type is cofinitely sensitive, (iv) any syndetically transitive, infinite subshift is syndetically sensitive, (v) no Sturmian subshift is cofinitely sensitive, (vi) we construct a transitive, sensitive map which is not syndetically sensitive.

Keywords: Sensitivity, transitivity, minimal map, interval map, syndetic set, subshift of finite type, Sturmian subshift, periodic point, recurrent point.

MSC 2000 primary: 54H20, 37B10.

1 Introduction

Sensitive dependence on initial conditions, or *sensitivity* for short, is a key ingredient of *chaos* for dynamical systems, see [1, 11, 13, 23]. For us, a dynamical system is a pair (X, f) where X is a compact metric space and $f : X \to X$ is a continuous map. Roughly speaking, a dynamical system is sensitive if given any region of the phase space X, there exist two points in the region and a time unit $n \in \mathbb{N}$ such that the n^{th} iterates of the two points under the map f are 'significantly separated'. The 'largeness' of the set of all $n \in \mathbb{N}$ where this 'significant separation' or 'sensitivity' happens, can be thought of as a measure of how sensitive the system is. For instance, if this set turns out to be rather 'thin' with arbitrarily large gaps between consecutive entries, then one has some excuse for treating the system as practically non-sensitive!

We restrict ourselves to the consideration of two simple notions of 'largeness': *syndeticity* and *cofiniteness*. We will see that these two notions are important in measuring the strength of sensitivity. Consider the following two natural questions about a sensitive dynamical system:

(i) Does 'sensitivity happen' at regular intervals of time?

(ii) Does 'sensitivity happen' for every unit of time after some stage?

Formulated mathematically, using the definitions provided in this article, these questions translate respectively into (i) is the system *syndetically sensitive*? (ii) is the system *cofinitely sensitive*? And this article is a preliminary investigation about these two notions. Introducing such stronger notions of sensitivity is further justified when we observe that:

(i) many familiar examples satisfy these stronger forms of sensitivity,

(ii) nice results can be found about them, and

(iii) for *transitivity*, another ingredient of *chaos*, stronger forms of the same vein are well-studied in the literature (eg: mixing = 'cofinite transitivity'), see also [2, 12, 15, 16, 17].

In passing, we remark that a stronger form of sensitivity called *Li-Yorke sensitivity* appeared recently in [3]. *Li-Yorke sensitivity* is formulated in terms of special type of individual points in the system. But our focus is not on individual points. We are interested in the 'set of times' where 'sensitivity happens', and in finding out how large this set is.

The main results are the following:

(i) Any syndetically transitive, non-minimal map is syndetically sensitive (this improves the results of [1], [4] that sensitivity is redundant in Devaney's definition of chaos) [Theorem 1].

(ii) Any sensitive map of $[0,1]$ is cofinitely sensitive [Theorem 2].

(iii) Any sensitive subshift of finite type is cofinitely sensitive [Theorem 3].

(iv) Any syndetically transitive, infinite subshift is syndetically sensitive [Theorem 4].

(v) No Sturmian subshift is cofinitely sensitive [Theorem 5].

(vi) We construct a transitive, sensitive map which is not syndetically sensitive [Section 8].

We shall provide more remarks as we go along. First let us have:

2 Basic Definitions

We have already defined a dynamical system. If (X, f) is a dynamical system and if $x \in X$, then the **orbit** of x, denoted by $O_f(x)$ is $\{x, f(x), f^2(x), f^3(x), \ldots\}$. Here f^n stands for the n-fold self-composition of f. If $f^n(x) = x$ for some $n \in \mathbb{N}$, we say that x is a **periodic point** for f. The set of periodic points of f will be denoted by $P(f)$. An element $x \in X$ is called a **recurrent point** for f if for some increasing sequence (n_k), $f^{n_k}(x) \to x$.

For $A \subset \mathbb{N}$, we say that A is **cofinite** if $\mathbb{N} \setminus A$ is finite, A is **thick** if A contains arbitrarily large blocks of consecutive numbers, and that A is **syndetic** if $\mathbb{N} \setminus A$ is not thick. Note that A is syndetic iff the following holds: A is infinite, and if A is written as $A = \{a_1 < a_2 < a_3 < \cdots\}$ then there exists $M \in \mathbb{N}$ such that $a_n - a_{n-1} < M$ for every $n \in \mathbb{N}$, where $a_0 := 0$. For a syndetic A, any such M will be referred to as a *bound for the gaps in A*.

For $A \subset \mathbb{N}$, the **upper density** of A is defined as

$$\overline{d}(A) := \limsup_{n \in \mathbb{N}} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}. \tag{1}$$

Similarly, the lower density of A is defined with lim sup replaced by lim inf. It may be noted that any syndetic set has positive lower density.

For a dynamical system, *transitivity* roughly means that within the phase space one can move from any region to any other region by iteration. *Mixing* and *syndetical transitivity* are stronger forms of transitivity with mixing implying syndetical transitivity. We define them precisely below. For a dynamical system (X, f) and subsets $U, V \subset X$, let

$$N_f(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}. \tag{2}$$

Let (X, f) be a dynamical system. We say that

(i) f is **transitive** if for every pair of nonempty open sets $U, V \subset X$, we have that $N_f(U, V)$ is nonempty.

(ii) f is **syndetically transitive** if for every pair of nonempty open sets $U, V \subset X$, we have that $N_f(U, V)$ is syndetic.

(iii) f is **mixing** if for every pair of nonempty open sets $U, V \subset X$, we have that $N_f(U, V)$ is cofinite.

We wish to formulate stronger notions of sensitivity in a similar manner. Let (X, f) be a dynamical system and let d be an admissible metric on X. According to the classical definition, f is **sensitive** if there exists $\delta > 0$ with the property that for any nonempty open set $U \subset X$, one can find $y, z \in U$ and $n \in \mathbb{N}$ such that $d(f^n(y), f^n(z)) > \delta$. We write this in a slightly different way. For $U \subset X$ and $\delta > 0$, let

$$N_f(U, \delta) = \{n \in \mathbb{N} : \text{ there exist } y, z \in U \text{ with } d(f^n(y), f^n(z)) > \delta\}. \tag{3}$$

Now, we say:

(i) f is **sensitive** if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is nonempty.

(ii) f is **syndetically sensitive** if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is syndetic.

(iii) f is **cofinitely sensitive** if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_f(U, \delta)$ is cofinite.

Here δ will be referred to as a *constant of sensitivity.* From the definition, cofinite sensitivity $\Longrightarrow$ syndetic sensitivity $\Longrightarrow$ sensitivity. It may also be noted that: (i) if f is sensitive, then f^n is sensitive for every $n \in \mathbb{N}$ (the constant of sensitivity may differ), and (ii) if f is sensitive with $\delta > 0$ as a constant of sensitivity, then $N_f(U, \delta)$ is infinite for every nonempty open set $U \subset X$.

A dynamical system (X, f) is a **minimal system** (or f is a **minimal map**) if the f-orbit of every $x \in X$ is dense in X. It is easy to see that (X, f) is a minimal system iff X has no proper, nonempty, closed f-invariant subset. Here, for a subset Y, f**-invariance** means $f(Y) \subset Y$. In a dynamical system (X, f), an element $x \in X$ is called a **minimal point** if the dynamical system $(\overline{O_f(x)}, f|_{\overline{O_f(x)}})$ is minimal. Note that every periodic point is a minimal point, and that every minimal point is a recurrent point.

It is a folklore result that $x \in X$ is a minimal point iff for every neighborhood U of x, the set $N_f(x, U) := \{n \in \mathbb{N} : f^n(x) \in U\}$ is syndetic (c.f. [12]). From this result, it is easy to deduce that a transitive map with a dense set of minimal points is syndetically transitive. In particular, every minimal map is syndetically transitive. We point out that there is another natural candidate for a syndetically transitive map. Any transitive map admitting an invariant Borel probability measure of full support is known to be syndetically transitive (c.f. [14]).

There is a natural situation where sensitivity implies syndetical sensitivity:

Proposition 1. *If f is sensitive and if the set of minimal points of f is dense in X, then f is syndetically sensitive with the same constant of sensitivity.*

Proof. Suppose that f is sensitive with constant $\delta > 0$. Let $U \subset X$ be a nonempty open set. We know that $k \in N_f(U, \delta)$ for some $k \in \mathbb{N}$. Since (X, f) has a dense set of minimal points, the product system $(X \times X, f \times f)$ also has a dense set of minimal points [2]. Therefore we can choose $y, z \in U$ such that (y, z) is a minimal point for $f \times f$ and such that $d(f^k(y), f^k(z)) > \delta$. Let $V, W \subset X$ be small enough open neighborhoods of $f^k(y)$ and $f^k(z)$, respectively, with $d(v, w) > \delta$ for every $v \in V$ and $w \in W$. In the system $(X \times X, f \times f)$, the open set $V \times W$ is a neighborhood of the minimal point $(f^k(y), f^k(z))$. Therefore, the set $N_{f\times f}((f^k(y), f^k(z)), V \times W)$ is syndetic. By the choice of V and W, we have $N_{f\times f}((f^k(y), f^k(z)), V \times W) + k \subset N_f(U, \delta)$. Therefore $N_f(U, \delta)$ is also syndetic. □

In particular, a sensitive system with a dense set of periodic points is syndetically sensitive - in fact, one can easily prove the stronger statement that $N_f(U, \delta)$ contains an infinite arithmetic progression.

3 Sensitivity from transitivity

In this section, we prove results of the following form: " a stronger form of transitivity + some condition $\Longrightarrow$ a stronger form of sensitivity". First, we state an elementary observation, whose obvious proof is omitted:

Proposition 2. *Let (X, f) be a mixing dynamical system. Then, for any positive $\delta < diam[X]$, f is cofinitely sensitive with δ as a constant of sensitivity.*

In [11], Devaney defined a dynamical system (X, f) to be **chaotic** if f is transitive, sensitive and if $P(f)$ is dense in X. It was soon observed in [4] that if X is not finite, f is transitive and if $P(f)$ is dense in X, then f is sensitive. In [1], this was improved by showing that if f is a transitive, non-minimal map with a dense set of minimal points, then f is sensitive. We improve this further:

Theorem 1. *Let (X, f) be a dynamical system. If f is syndetically transitive but not minimal, then f is syndetically sensitive.*

Proof. Let $a \in X$ be such that $O_f(a)$ is not dense in X. Let $b \in X \setminus \overline{O_f(a)}$, and put $\delta = \frac{1}{4} d(b, \overline{O_f(a)}) > 0$. Write $V = B(b, \delta)$. If $U \subset X$ is any nonempty open set, then $N_f(U, V)$ is syndetic, with say M_1 as a bound for the gaps. Choose an open set W around a such that $x \in W \Longrightarrow d(f^i(a), f^i(x)) < \delta$ for $i = 0, 1, \ldots, M_1$. Note that then $d(f^i(W), V) \geq 2\delta$ for $i = 0, 1, \ldots, M_1$, by the choice of δ. Now, $N_f(U, W)$ is also syndetic. Let M_2 be a bound for the gaps in $N_f(U, W)$. We show that $N_f(U, \delta)$ is syndetic with $M_1 + M_2$ as a bound for the gaps. Let $n \in \mathbb{N}$. Choose $j \in \{1, \ldots, M_2\}$ and $u \in U$ such that $f^{n+j}(u) \in W$. Then, by the choice of W, one has that for every $i \in \{1, \ldots, M_1\}$, $d(f^{n+j+i}(u), V) \geq 2\delta$. Choose $i \in \{1, \ldots, M_1\}$ and $u' \in U$ such that $f^{n+j+i}(u') \in V$. Then, for this particular i, we have $d(f^{n+j+i}(u), f^{n+j+i}(u')) \geq 2\delta > \delta$. Since $n \in \mathbb{N}$ is arbitrary and since $j + i \leq M_1 + M_2$, the argument is complete. □

Corollary 1. *For a syndetically transitive dynamical system, sensitivity implies syndetical sensitivity.*

Proof. If the system has a dense set of minimal points, use Proposition 1. Else, apply Theorem 1. □

If α is an irrational, then the isometry $x \mapsto e^{2\pi i\alpha}x$ on the unit circle is known as an **irrational rotation**. It is well-known that any irrational rotation is minimal, and hence syndetically transitive. This example shows that syndetical transitivity alone cannot imply sensitivity. However, we have the following sufficient condition:

Proposition 3. *Let (X, f) be a dynamical system. If f is syndetically transitive and if*

$$\inf_{n \in A} \sup_{x \in X} d(x, f^n(x)) > \delta > 0$$

for some thick set $A \subset \mathbb{N}$, then f is syndetically sensitive with δ as a constant of sensitivity.

Proof. Let $U \subset X$ be nonempty open. Since $N_f(U, U)$ is syndetic and A is thick, there exists $n \in N_f(U, U) \cap A$. Put $W = U \cap f^{-n}(U)$. Then, W is nonempty and open. Since $n \in A$, by hypothesis we can find $x \in X$ such that $d(x, f^n(x)) > \delta$. Let V be an open set containing x such that $y \in V$ implies $d(y, f^n(y)) > \delta$. Now, consider the syndetic set $N_f(W, V)$. We claim that it is contained in $N_f(U, \delta)$. Let $k \in N_f(W, V)$ and let $a \in W \subset U$ be such that $f^k(a) \in V$. Then, $d(f^k(a), f^{k+n}(a)) > \delta$ by the choice of V. So if we put $b = f^n(a)$, then $d(f^k(a), f^k(b)) > \delta$. Also, $b \in f^n(W) \subset U$. Therefore $k \in N_f(U, \delta)$, and this establishes the claim. □

Corollary 2. *Let (X, f) be a syndetically transitive system. Suppose that there exist two distinct points $x, y \in X$ and a thick set $\{n_k : k \in \mathbb{N}\}$ with $\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = 0$. Then f is syndetically sensitive.*

Proof. Choose a positive $\delta < \frac{1}{3}d(x, y)$. Let $k_0 \in \mathbb{N}$ be such that $d(f^{n_k}(x), f^{n_k}(y)) < \delta$ for every $k \geq k_0$. Then, for each $k \geq k_0$, by triangle inequality we have that $d(x, f^{n_k}(x)) > \delta$ or $d(y, f^{n_k}(y)) > \delta$. So Proposition 3 applies with $A = \{n_k \in \mathbb{N} : k \geq k_0\}$. □

4 Sensitivity for continuous maps on $[0, 1]$

In this section, we show that every sensitive map $f : [0, 1] \to [0, 1]$ is cofinitely sensitive. This is partially due to the abundance of periodic points.

By an interval, we always mean an interval of positive length (it may be closed, open, or neither, but never a singleton). In the sequel we will use the following simple fact without explicitly stating it: if $f : [0, 1] \to [0, 1]$ is sensitive, then for every interval $J \subset [0, 1]$ and every $n \in \mathbb{N}$, $f^n(J)$ is also an interval.

Lemma 1. *Let $L \subset \mathbb{R}$ be a compact interval and $f : L \to L$ be sensitive. Then, the closure of the set of periodic points of f contains an interval.*

Proof. From Proposition 2.2.5 (Blokh) of [23], it can be deduced that there exist a closed interval $J \subset L$ and an $n \in \mathbb{N}$ such that J is f^n-invariant and $f^n|_J : J \to J$ is transitive. But a transitive map of a closed interval has a dense set of periodic points [5]. Thus $J \subset \overline{P(f^n)} = \overline{P(f)}$. □

Theorem 2. *Let $f : [0,1] \to [0,1]$ be sensitive. Then f is cofinitely sensitive.*

Proof. Let $\delta > 0$ be a constant of sensitivity for f. Choose finitely many periodic points $x_1, \ldots, x_r \in [0,1]$ such that for any interval $J \subset \overline{P(f)}$ with $diam[J] > \delta$, we have that $|J \cap \{x_1, \ldots, x_r\}| \geq 2$. Let $\alpha = \min\{|x_i - x_j| : 1 \leq i < j \leq r\} > 0$ and let $k \in \mathbb{N}$ be such that $f^k(x_i) = x_i$ for $1 \leq i \leq r$. Let $\beta > 0$ be such that for every $x, y \in [0,1]$, $|x - y| \leq \beta$ implies $|f^i(x) - f^i(y)| < \alpha$ for $0 \leq i \leq k$. We claim that f is cofinitely sensitive with β as a constant of sensitivity.

Let $J \subset [0,1]$ be any interval. Since $diam[f^n(J)] > \delta$ for infinitely many n, it is easy to see that $\bigcup_{n=0}^{\infty} f^n(J)$ has only finitely many connected components. Hence the same is true for $\overline{\bigcup_{n=0}^{\infty} f^n(J)}$. Therefore, one can find a connected component L, which must be a closed interval, of $\overline{\bigcup_{n=0}^{\infty} f^n(J)}$ and an $n \in \mathbb{N}$ such that $f^n(L) \subset L$. Then, $f^n|_L : L \to L$ is sensitive (with some constant of sensitivity). By Lemma 1, $\overline{P(f^n|_L)}$ contains an interval. This implies that, for some $s \in \mathbb{N}$, $f^s(J) \cap \overline{P(f)}$ contains an interval, say K. Now, for some $t \in \mathbb{N}$, $diam[f^t(K)] > \delta$. But the interval $f^t(K)$ is contained in $\overline{P(f)}$ as $\overline{P(f)}$ is f-invariant. Therefore, $|f^t(K) \cap \{x_1, \ldots, x_r\}| \geq 2$. As a consequence, $diam[f^{t+j}(K)] > \beta$ for every $j \in \mathbb{N}$, by the choice of β. Hence, $diam[f^{s+t+j}(J)] > \beta$ for every $j \in \mathbb{N}$. That is, $s + t + \mathbb{N} \subset N_f(J, \beta)$. □

Thus all sensitive maps of $[0,1]$ exhibit a very strong form of sensitivity. We know [5] that transitivity implies sensitivity on $[0,1]$. Therefore, by the above Theorem, all transitive maps on $[0,1]$ are cofinitely sensitive. In the rest of the article, we distinguish sensitivity, syndetical sensitivity and cofinite sensitivity using subclasses of dynamical systems known as *subshifts*.

5 Subshifts

By a *subshift* we always mean a *one-sided subshift*. We will give all the necessary definitions related to *subshifts*. But we will be rather brief in our arguments. If the reader is not quite familiar with the standard arguments used in the theory of *subshifts*, [6, 18, 19] are some of the good sources to seek help from.

For an integer $m \geq 2$, let $\Sigma_m = \{x = (x_n)_{n=0}^{\infty} : x_n \in \{1, 2, \ldots, m\}$ for every $n\}$. Considering $\{1, 2, \ldots, m\}$ as a discrete space and putting product topology on Σ_m, we see that Σ_m is homeomorphic to the Cantor set. An admissible metric on Σ_m is given by

$$d(x, y) = \sum_{\{n \geq 0 :\, x_n \neq y_n\}} 2^{-n}, \text{ for } x, y \in \Sigma_m. \tag{4}$$

For $n \in \mathbb{N}$, if (a nonempty) $w \in \{1, \ldots, m\}^n$, we say w is a **word** of **length** n, and we write $|w| = n$. If w, v are words and if w appears in v we say w is a **subword** of v (eg: 1121, 212 are subwords of 112122 but 1122 is not a subword of 112122), and we write $w \sqsubset v$. Similarly for $x \in \Sigma_m$, we write $w \sqsubset x$ if w appears in x as a single block.

For a word w, and $n \in \mathbb{N}$, we define $w^n = \underbrace{ww \cdots w}_{n-\text{times}}$. In particular, 1^n is the word of length n consisting entirely of 1's. If $x \in \Sigma_m$ and if $i \leq j$ are non-negative integers, $x_{[i,i+j]}$ will denote the word $x_i x_{i+1} \cdots x_{i+j}$.

Observe that for $x, y \in \Sigma_m$, and $n \geq 0$,

$$d(x,y) < 2^{-n} \Rightarrow x_{[0,n]} = y_{[0,n]} \Rightarrow d(x,y) \leq 2^{-n}. \tag{5}$$

Let $\sigma : \Sigma_m \to \Sigma_m$ be the **shift map** defined by $(x_0 x_1 x_2 \cdots) \mapsto (x_1 x_2 \cdots)$. This is clearly continuous. If $X \subset \Sigma_m$ is a nonempty, closed σ-invariant set, then the dynamical system $(X, \sigma|_X)$ will be called a **subshift**. Instead of $(X, \sigma|_X)$, we will only write (X, σ) when there is no scope for confusion. One common way of producing a subshift is to take $X = \overline{O_\sigma(x)}$ for some $x \in \Sigma_m$, but not all subshifts arise in this fashion. Two well-known properties regarding subshifts are given below. We provide a short proof for the sake of completeness.

Proposition 4. *Let (X, σ) be a subshift.*
(i) (X, σ) is sensitive iff X is infinite and has no isolated points.
(ii) Suppose that $X = \overline{O_\sigma(x)}$ for some $x \in \Sigma_m$. If x is a recurrent point but not a periodic point for σ, then the system (X, σ) is transitive and sensitive.

Proof. (i) It is clear that a finite space or a space having an isolated point cannot admit any sensitive map - if $\{x\}$ is open, then there is no question of finding two distinct points $y, z \in U = \{x\}$. Conversely, assume that X is infinite and has no isolated points. Therefore, any nonempty open set $U \subset X$ is infinite, and so we can find $y, z \in U$ and $n \in \mathbb{N}$ such that $y_n \neq z_n$. This is same as saying $[\sigma^n(y)]_0 \neq [\sigma^n(z)]_0$. Hence $d(\sigma^n(y), \sigma^n(z)) \geq 1$.
(ii) Since x is a recurrent point for σ, the system (X, σ) is transitive. Since x is not a periodic point, X is not finite. An infinite space admitting a transitive map cannot have any isolated points. Now use (i). □

It is convenient to express syndetic sensitivity and cofinite sensitivity of a subshift in terms of words. For a subshift (X, σ), let $W(X) = \{w \in \bigcup_{n=1}^{\infty} \{1, \ldots, m\}^n : w \sqsubset y \text{ for some } y \in X\}$. In a particular case, when $X = \overline{O_\sigma(x)}$, one can see that $W(X)$ is simply the collection of all words w appearing in x. For $w \in W(X)$, let $U_w = \{y \in X : y \text{ starts with } w\}$. Then, U_w is a nonempty set which is both closed and open in X. Moreover, the collection $\{U_w : w \in W(X)\}$ forms a countable base for the topology on X. These are standard facts. For $w \in W(X)$, we write

$$D(w) = \{n \in \mathbb{N} : \text{there exist } y, z \in U_w \text{ such that } y_n \neq z_n\}. \tag{6}$$

Using the inequality 5, it can be deduced that:

Lemma 2. *Let (X, σ) be a subshift. Then,*
(i) (X, σ) is syndetically sensitive iff $D(w)$ is syndetic for every $w \in W(X)$.
(ii) (X, σ) is cofinitely sensitive iff there exists a natural number M such that $\bigcup_{l=1}^{M} [D(w) + l]$ is cofinite for every $w \in W(X)$.

The set $D(w)$ can be expressed purely in terms of words. We leave it as an exercise to check the following:

$$D(w) = \{|w| + k : k \geq 0, \exists\, 1 \leq i < j \leq m, \exists\, u, v \in \{1, \ldots, m\}^k \text{ such that } wui, wvj \in W(X)\}. \tag{7}$$

6 Sensitivity for subshifts of finite type

One important class of subshifts is *subshifts of finite type*. There are some similarities between *subshifts of finite type* and dynamical systems defined on [0,1]. For example, for both the classes, (i) transitivity implies denseness of periodic points, and (ii) total transitivity implies mixing (for *subshifts of finite type*, (ii) was observed in [21]; for the other results, see for instance, [8, 9, 23]). The main result of this section, Theorem 3, is another instance of similarity; compare with Theorem 2.

Let P be an $m \times m$ matrix with entries in $\{0, 1\}$. We will denote the $(i, j)^{th}$ entry of the matrix P by $P(i, j)$. Let $X_P = \{x \in \Sigma_m : P(x_n, x_{n+1}) = 1 \text{ for every } n \geq 0\}$. One can verify that (X_P, σ) is indeed a subshift. A subshift given in this way (by a finite data presented by a square matrix of 0's and 1's) is called a **subshift of finite type**. Note the following relation: for any $n \in \mathbb{N}$, the $(i, j)^{th}$ entry of the matrix P^n is simply the cardinality of the finite set $\{w \in W(X_P) : |w| = n + 1, w \text{ starts with } i \text{ and ends with } j\}$. This allows us to deduce that:

Lemma 3. *Let (X_P, σ) be a subshift of finite type. Then, X is an infinite space without isolated points iff for each $i \in \{1, \ldots, m\}$ there exists $n \in \mathbb{N}$ such that the i^{th} row of the matrix P^n has atleast two positive entries.*

Theorem 3. *Let (X_P, σ) be a subshift of finite type. If (X_P, σ) is sensitive, then it is cofinitely sensitive.*

Proof. For $i \in \{1, \ldots, m\}$, let $n_i \in \mathbb{N}$ be such that the i^{th} row of the matrix P^{n_i} has atleast two positive entries. The existence of n_i is assured by Proposition 4(i) and Lemma 3. Let M be a natural number greater than all the n_i's. Then, $D(i) \cap \{1, \ldots, M\} \neq \emptyset$ for $1 \leq i \leq m$. We claim that $\bigcup_{l=1}^{M} [D(w) + l]$ is cofinite for every $w \in W(X_P)$.

This is a consequence of a special property of subshifts of finite type: if (X_P, σ) is a subshift of finite type, then $wi, iu \in W(X_P)$ implies $wiu \in W(X_P)$ for words w, u and $i \in \{1, \ldots, m\}$. This implies that $|w| + D(i) \subset D(wi)$ for every word w and $i \in \{1, \ldots, m\}$ with $wi \in W(X_P)$. We use it to show that for every $w \in W(X_P)$ and every $k \geq 0$, $D(w) - |w|$ intersects $\{k + 1, k + 2, \ldots, k + M\}$.

Consider $w \in W(X_P)$ and $k \geq 0$. Let $y \in U_w$ and let $i = y_{|w|+k}$. From the previous paragraphs, we know that $D(i) \cap \{1, \ldots, M\} \neq \emptyset$ and $|w| + k + D(i) \subset D(y_{[0,\,|w|+k]})$. But $D(y_{[0,\,|w|+k]}) \subset D(w)$. It follows that $[D(w) - |w|] \cap \{k + 1, k + 2, \ldots, k + M\} \neq \emptyset$.

Therefore, $D(w) - |w|$ is a syndetic set with gaps bounded by M, for every $w \in W(X_P)$. Hence the claim. To finish the proof, apply Lemma 2(ii). □

7 Syndetically sensitive but not cofinitely sensitive

Not every sensitive subshift is cofinitely sensitive. In this section, we show that the so called *Sturmian subshifts* are syndetically sensitive but not cofinitely sensitive. For a real number r we will denote the fractional part of r by $frac(r)$. Let $\alpha \in (0, 1)$ be an irrational. Define an element $x^\alpha \in \Sigma_2$ by the following rule:

$$x_n^\alpha = \begin{cases} 1, \text{ if } frac(n\alpha) \in [0, 1-\alpha), \\ 2, \text{ if } frac(n\alpha) \in [1-\alpha, 1). \end{cases}$$

Put $X_\alpha = \overline{O_\sigma(x^\alpha)}$. Then the subshift (X_α, σ) is called a **Sturmian subshift**. It is known that if (X_α, σ) is a Sturmian subshift, then it is minimal with X_α infinite (c.f. [18]). We would like to say that (X_α, σ) is syndetically sensitive. For this purpose, we prove a general result:

Theorem 4. *Let (X, σ) be a syndetically transitive subshift with X infinite. Then (X, σ) is syndetically sensitive.*

Proof. Since (X, σ) is transitive and X is infinite, X cannot have any isolated points. Hence (X, σ) is sensitive by Proposition 4. Now, by Corollary 1, (X, σ) is syndetically sensitive. □

Corollary 3. *Any Sturmian subshift is syndetically sensitive.*

Next, we wish to show that no Sturmian subshift is cofinitely sensitive. For an irrational $\alpha \in (0, 1)$, let $f_\alpha : [0, 1) \to [0, 1)$ be the map $t \mapsto frac(t + \alpha)$. Note that $x_n^\alpha = 1$ or 2 according as $f_\alpha^n(0) \in [0, 1-\alpha)$ or $f_\alpha^n(0) \in [1-\alpha, 1)$. Therefore, in the case of a Sturmian subshift (X_α, σ), we can translate the criterion for cofinite sensitivity given in terms of $D(w)$'s in Lemma 2(ii) to a criterion involving f_α and open subintervals of $(0, 1)$. For an open interval $J \subset (0, 1)$, let

$$D_\alpha(J) = \{n \in \mathbb{N} : f_\alpha^n(J) \text{ intersects both } [0, 1-\alpha) \text{ and } [1-\alpha, 1)\}. \tag{8}$$

The following can be directly verified.

Lemma 4. *A Sturmian subshift (X_α, σ) is cofinitely sensitive iff there exists a natural number M such that $\bigcup_{l=1}^{M} [D_\alpha(J) + l]$ is cofinite for every open interval $J \subset (0, 1)$.*

Since f_α is a translation, it can also be observed that

$$D_\alpha(J) = \{n \in \mathbb{N} : f_\alpha^n(J) \cap \{0, 1-\alpha\} \neq \emptyset\}. \tag{9}$$

It is a consequence of *Weyl's Theorem of uniform distribution mod 1*, which can be found in [10], that for any $t \in [0, 1)$ and any open interval $J \subset (0, 1)$, the upper density of the set $\{n \in \mathbb{N} : t \in f_\alpha^n(J)\}$ is equal to the diameter (length) of J. Therefore, from the expression 9, it follows that the upper density of $D_\alpha(J)$ is atmost two times the diameter of J. Now, given any natural number M, choose an open interval $J \subset (0, 1)$ such that $diam[J] < \frac{1}{2M}$. Then, the upper density of the set $\bigcup_{l=1}^{M} [D_\alpha(J) + l]$ is strictly less than 1 and hence this set cannot be cofinite. This argument establishes that

Theorem 5. *No Sturmian subshift is cofinitely sensitive.*

Now, we have a good picture regarding sensitivity of minimal systems:

Theorem 6. *For a minimal dynamical system (X, f), consider the following statements:*
(i) The family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous.

(ii) f is sensitive.

(iii) f is syndetically sensitive.

(iv) f is cofinitely sensitive.

Then, exactly one of (i), (ii) holds. Statements (ii) and (iii) are equivalent; (iv) is strictly stronger than (iii). Also, there are examples of minimal systems satisfying each of (i) and (iv).

Proof. The result that exactly one of (i), (ii) holds for a minimal system, can be found in [1]. The equivalence of (ii) and (iii) follows from Corollary 1. That (iv) is strictly stronger than (iii) is witnessed by Sturmian subshifts. Irrational rotations are minimal systems satisfying (i). As an example for (iv), we can consider any minimal map which is mixing (see, for instance, [22]). □

8 Sensitive but not syndetically sensitive

In this section we construct a transitive, sensitive subshift which is not syndetically sensitive. Let (n_k) be an increasing sequence of natural numbers. We will give more information about the values of n_k later. Recursively define a sequence $w_1, w_2, \ldots$ of words over $\{1, 2\}$ as follows:

$$w_1 = 2, \;\; w_2 = 1^{n_1}, \;\; w_{2k+1} = w_1 w_2 w_3 \cdots w_{2k}, \;\; w_{2k+2} = 1^{n_{k+1}}. \tag{10}$$

Put $x = w_1 w_2 w_3 \cdots \in \Sigma_2$. That is,

$$x = \underbrace{21^{n_1}}_{w_1 w_2}\underbrace{21^{n_1}}_{w_3}\underbrace{1^{n_2}}_{w_4}\underbrace{21^{n_1}21^{n_1+n_2}}_{w_5}\underbrace{1^{n_3}}_{w_6}\underbrace{21^{n_1}21^{n_1+n_2}21^{n_1}21^{n_1+n_2+n_3}}_{w_7}\underbrace{1^{n_4}}_{w_8}\cdots. \tag{11}$$

Then, x is clearly a recurrent point for the shift map σ. Since the sequence (n_k) is increasing, x is not a periodic point for σ. Therefore, if we put $X = \overline{O_\sigma(x)}$, then (X, σ) is a transitive, sensitive subshift, by Proposition 4(ii). We show that if (n_k) increases with 'sufficient speed', then (X, σ) is not syndetically sensitive. Our argument is based on three elementary observations about x. Write $s_k = \sum_{i=1}^{k} n_i$ for $k \in \mathbb{N}$. The reader may verify the following:

(i) If $21^m 2 \sqsubset x$ for some $m \geq 0$, then $m > 0$ and $m = s_k$ for some $k \in \mathbb{N}$.

(ii) For every $k \in \mathbb{N}$, the word $21^{s_k}2$ appears in x infinitely often.

(iii) For any word w, if $21^{s_k}2\, w\, 21^{s_k}2 \sqsubset x$, then $1^{s_j} \sqsubset w$ for some $j > k$.

Since (X, σ) is a transitive subshift generated by a recurrent point x, $W(X)$ is the collection of all words appearing in x. Therefore, if we put

$$A = \{n - n' : 0 \leq n' < n, \;\; x_n = x_{n'} = 2\}, \tag{12}$$

then it is clear that $D(w)$ is contained in a translate of A for any $w \in W(X)$ with $2 \sqsubset w$. In fact, if 2 appears in the j^{th} position of the word w, then $D(w) \subset A + j - 1$. Hence, to establish that (X, σ) is not syndetically sensitive, it suffices to show that the set A is not syndetic.

Using the observation (i), it is not difficult to see that any element $a \in A$ is of the form $a = l + \sum_{i=1}^{k} a_i s_i$, where a_i's are non-negative integers with $\sum_{i=1}^{k} a_i = l$. We may assume that $a_k > 0$. Then, by (iii),

$a_k = 1$. Also, (iii) implies that $a_i \leq 1 + \sum_{j=i+1}^{k} a_j$ for $i < k$. Hence, inductively $a_{k-i} \in \{0, 1, \ldots, 2^i\}$ for $i = 1, \ldots, k-1$. Now, for natural numbers $l \leq k$, define $B_{k,l}$ to be the collection of all k-tuples $(a_1, \ldots, a_k)$ of non-negative integers satisfying $\sum_{i=1}^{k} a_i = l$, $a_k = 1$, and $a_i \leq 1 + \sum_{j=i+1}^{k} a_j$ for $i < k$. Then,

$$|B_{k,l}| \leq \prod_{i=1}^{k-1} (1 + 2^i) \leq \prod_{i=2}^{k} 2^i \leq 2^{k(k+1)/2}. \tag{13}$$

Let

$$B = \{l + \sum_{i=1}^{k} a_i s_i : k, l \in \mathbb{N},\ l \leq k \text{ and } (a_1, \ldots, a_k) \in B_{k,l}\}. \tag{14}$$

Since $A \subset B$, it is enough to show that B is not syndetic.

Now, $B \cap \{1, 2, \ldots, s_k\} \subset \{l + \sum_{i=1}^{j} a_i s_i : 1 \leq l \leq j \leq k, (a_1, \ldots, a_j) \in B_{j,l}\}$. Therefore,

$$|B \cap \{1, 2, \ldots, s_k\}| \leq \sum_{1 \leq l \leq j \leq k} |B_{j,l}| \leq \sum_{1 \leq l \leq j \leq k} 2^{k(k+1)/2} \leq k^2 2^{k(k+1)/2} \leq 2^{k^2}, \tag{15}$$

where the last inequality holds for all $k \geq 4$. Hence, if the sequence (n_k) is such that $2^{k^2}/s_k \to 0$ as $k \to \infty$, then B has lower density zero and hence B cannot be syndetic. This completes the argument.

It is also clear that if the sequence (n_k) is such that $2^{(k+1)^2}/s_k \to 0$ as $k \to \infty$, then B has (upper) density zero. Thus we have:

Theorem 7. *Let (n_k), x, (X, σ) be as in this section. Let $s_k = \sum_{i=1}^{k} n_i$. Then, (X, σ) is transitive, sensitive, and has the following property: if $2^{(k+1)^2}/s_k \to 0$ as $k \to \infty$, then $D(w)$ has zero density for every $w \in W(X)$ with $2 \sqsubset w$. (So, in some sense, the sensitive system (X, σ) is very close to a non-sensitive system.)*

By the way, note that the dynamical system constructed above serves also as an example of a transitive system which is not syndetically transitive, by Corollary 1.

9 Concluding remarks and questions

1. It is interesting to note that any Sturmian subshift, which we have proved to be syndetically sensitive, is an *almost one-one extension* (see [2] for a definition and for more details) of an irrational rotation of the unit circle, which is an isometry, and any isometry is far from being sensitive. However, we do not know whether there exists a cofinitely sensitive system which is an *almost one-one extension* of an isometry.
2. *Topological entropy* [20] is often used to measure the complexity of a dynamical system. There exists dynamical systems which are mixing but having zero *topological entropy* [24]. Since mixing implies cofinite sensitivity, it follows that cofinite sensitivity need not imply positive *topological entropy.* But we may investigate another point. For a dynamical system, *Li-Yorke chaos* is known to be weaker than having positive *topological entropy* [7]. Does cofinite sensitivity imply *Li-Yorke chaos*?
3. In view of Corollary 2, *asymptotic relation* along 'large' subsets of $\mathbb{N}$ seems to be a topic inviting further investigation, see also [16].

4. Characterize syndetical sensitivity for a transitive dynamical system.

5. Find some conditions implying cofinite sensitivity for a dynamical system.

6. In this article, we have considered only syndetic and cofinite sensitivity. But it is clear that other stronger forms of sensitivity can be defined in a similar fashion. For instance, instead of syndeticity and cofiniteness, we may demand any of the following properties for $N_f(U, \delta)$:

(i) it is thick,

(ii) it contains a multiple of k for every $k \in \mathbb{N}$,

(iii) it has positive upper density,

(iv) it contains an infinite arithmetic progression, etc..

All these notions will be weaker than cofinite sensitivity, but their status compared with syndetic sensitivity is not always clear. A slightly different way in which we can formulate a stronger form of sensitivity is: we define (X, f) to be **multi-sensitive** if there exists $\delta > 0$ with the property that for every $k \in \mathbb{N}$ and nonempty open sets $U_1, \ldots, U_k \subset X$, we have that $\bigcap_{i=1}^{k} N_f(U_i, \delta) \neq \emptyset$. It is not difficult to show that

(i) Cofinite sensitivity $\Longrightarrow$ multi-sensitivity $\Longrightarrow$ $N_f(U, \delta)$ is thick for every nonempty open U.

(ii) If $f \times f$ is transitive (this is known as **weak mixing**), then f is multi-sensitive.

All these suggest that a study of stronger forms of sensitivity formulated in terms of various 'large' subsets of $\mathbb{N}$, has rich possibilities.

Acknowledgement: The detailed remarks of one referee were helpful in correcting many mistakes in the original draft of the paper, while the remarks of the other referee were helpful in coming up with Corollary 1 and Theorem 6.

References

[1] E.Akin, J.Auslander and K.Berg, *When is a transitive map chaotic?* Convergence in ergodic theory and probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., 5, de Gruyter, Berlin, (1996), 25-40.

[2] E.Akin and E.Glasner, *Residual properties and almost equicontinuity*, J. d'Analyse Mathematique, 84, (2001), 243-286.

[3] E.Akin and S.Kolyada, *Li-Yorke sensitivity*, Nonlinearity, 16, (2003), 1421-1433.

[4] J.Banks, J.Brooks, G.Cairns, G.Davis and P.Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly, 99, (1992) 332-334.

[5] R.Berglund and M.Vellekoop, *On intervals, transitivity = chaos*, Amer. Math. Monthly, 101, (1994), 353-355.

[6] F.Blanchard *et al* (Eds.), *Topics in Symbolic Dynamics and Applications*, Cambridge Univ. Press, (2000).

[7] F.Blanchard, E.Glasner, S.Kolyada and A.Maass, *On Li-Yorke pairs*, J. fur die reine und angewandte Mathematik (Crelle's J.), 547, (2002), 51-68.

[8] L.S.Block and W.A.Coppel, *Dynamics in One Dimension*, Lecture Notes in Mathematics, 1513, Springer, (1992).

[9] M.Brin and G.Stuck, *Introduction to Dynamical Systems*, Cambridge Univ. Press, (2002).

[10] K.Chandrasekharan, *Introduction to Analytic Number Theory*, Springer-Verlag, (1968).

[11] R.L.Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd edn., Addison Wesley, (1989).

[12] H.Furstenberg, *Disjointness in ergodic theory, minimal sets and a problem in Diophantine approximation*, Math. Sys. Th., 1, (1967), 1-49.

[13] E.Glasner and B.Weiss, *Sensitive dependence on initial conditions*, Nonlinearity, 6, (1993), 1067-1075.

[14] E.Glasner and B.Weiss, *Locally equicontinuous dynamical systems*, Colloq. Math., 84-85, (2000), 345-361.

[15] W.Huang and X.Ye, *An explicit scattering, non-weakly mixing example and weak disjointness*, Nonlinearity, 15, (2002), 849-862.

[16] W.Huang, S.Shao and X.Ye, *Mixing and proximal cells along sequences*, Nonlinearity, 17, (2004), 1245-1260.

[17] W.Huang and X.Ye, *Topological complexity, return times and weak disjointness*, Ergod. Th. & Dyn. Sys., 24, (2004) 825-846.

[18] P.Kurka, *Topological and Symbolic Dynamics*, Cours Spécialisés, 11, SMF, (2003).

[19] D.Lind and B.Marcus, *Symbolic Dynamics and Coding*, Cambridge Univ. Press (1995).

[20] M.Misiurewicz, *On Bowen's definition of topological entropy*, Discrete and Cont. Dyn. Sys., 10, (2004) 827-833.

[21] T.K.S.Moothathu, *Studies in topological dynamics with emphasis on cellular automata*, PhD thesis, University of Hyderabad, (2006).

[22] K.E.Petersen, A topologically strongly mixing symbolic minimal set, Trans. Amer. Math. Soc., 148, (1970), 603-612.

[23] S.Ruette, *Chaos for continuous interval maps - a survey of relationship between the various sorts of chaos*, (book - preprint - available electronically).

[24] B.Weiss, *Topological transitivity and ergodic measures*, Math. Sys. Th., 5, (1971), 71-75.