Quantitative views of recurrence and proximality

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Abstract

In a topological dynamical system with a dense set of recurrent points, we investigate whether there are "plenty" of points whose recurrence is "fast". Depending upon how we make our query precise, we get affirmative as well as negative answers. We carry out a similar study about proximal pairs; that is, for "most" proximal pairs of points, how fast the distance between the corresponding terms in the two orbits can go to zero. For instance, we show that if $f : [0, 1] \to [0, 1]$ is a continuous map having a periodic point whose period is not a power of 2, then for every function $\phi : \mathbb{N} \to \mathbb{N}$, there is an uncountable scrambled set $S \subset [0, 1]$ for f satisfying the extra property that $\liminf_{n\to\infty} \phi(n)|f^n(x) - f^n(y)| = 0$ for all $x, y \in S$. We also provide characterizations of weak mixing and mixing for a topological dynamical system in terms of proximality of orbits to arbitrary sequences in the phase space.

Key words: Recurrent point, proximal pair, transitivity, weak mixing, lower density.
MSC 2000: 54H20.

1 Introduction

The notion of recurrence holds a central place in the theory of dynamical systems, see for example [9, 10]. The topological version of Poincare's recurrence theorem (c.f. [7]) says that if a continuous map $f : X \to X$ of a compact metric space (X, d) admits an invariant Borel probability measure of full support, then the set $R(f)$ of recurrent points of f is residual in X. Note that $x \in X$ is called a recurrent point for f if

$$\liminf_{n\to\infty} d(x, f^n(x)) = 0. \tag{1}$$

Now one might wonder whether it is possible to get a stronger conclusion, say,

$$\liminf_{n\to\infty} [n^3 d(x, f^n(x))] = 0 \tag{2}$$

for a residual set of points. Here we are demanding a certain fastness for the rate of recurrence. More generally, given a function $\phi : \mathbb{N} \to \mathbb{N}$, we may define

$$R(f, \phi) = \{x \in X : \liminf_{n\to\infty} \phi(n) d(x, f^n(x)) = 0\}, \tag{3}$$

and we may ask whether it is residual. We will see that it is indeed so for every $\phi : \mathbb{N} \to \mathbb{N}$, iff f has a dense set of periodic points (Theorem 2).

There are other ways to formulate the fastness of recurrence, and one such way is by considering the lower density of the set of return times to any neighborhood. A little more generally, if $\phi : \mathbb{N} \to \mathbb{N}$ is any strictly increasing function, and $\epsilon > 0$, define

$$R_L(f, \phi, \epsilon) = \{x \in X : \liminf_{n\to\infty} \frac{|\{1 \le j \le \phi(n) : d(x, f^j(x)) \le \epsilon\}|}{n} > 0\}, \tag{4}$$

and put

$$R_L(f, \phi) = \bigcap_{\epsilon>0} R_L(f, \phi, \epsilon). \tag{5}$$

We show that even though $R_L(f, \phi)$ is of full measure in the presence of an invariant Borel probability measure, it is not topologically "big" in general (Theorem 3). In particular, we will establish that if $f : [0, 1] \to [0, 1]$ is a continuous map with a dense set of recurrent points and if $f^2 \neq Id$ on any subinterval of positive length, then $R_L(f, \phi)$ is of first category in $[0, 1]$ for any strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$ (Theorem 4).

Next, we consider the rate of recurrence in terms of suitable sets rather than points. If (X, T, μ) is a measure dynamical system, i.e., if $T : X \to X$ is a measure preserving transformation of a probability measure space (X, μ), and if $A \subset X$ is with $\mu[A] > 0$, we define

$$\beta_T(A, n) = \min\{k \in \mathbb{N} : \exists\, 1 \le j_1 < \cdots < j_n \le k \text{ with } \mu[A \cap T^{-j_1}(A) \cap \cdots \cap T^{-j_n}(A)] > 0\}. \tag{6}$$

Similarly, if (X, f) is a topological dynamical system with a dense set of recurrent points, and if $U \subset X$ is a nonempty open set, we define

$$\beta_f(U, n) = \min\{k \in \mathbb{N} : \exists\, 1 \le j_1 < \cdots < j_n \le k \text{ with } U \cap f^{-j_1}(U) \cap \cdots \cap f^{-j_n}(U) \neq \emptyset\}. \tag{7}$$

We show that, in a measure dynamical system, the growth rate of $\beta_T(A, n)$ with respect to n is linear (so that the recurrence is "fast"), and in a topological dynamical system, even when it is mixing, the growth rate of $\beta_f(U, n)$ with respect to n can be arbitrarily high (so that the recurrence is as "slow" as we want) (Theorems 5, 6).

In Section 6, we consider the notion of proximality. In [2], Akin and Kolyada proved that in a compact weak mixing system, the proximal cell of every point is a dense G_δ. We go one step further and show that indeed weak mixing and mixing can be characterized using the notion of proximality of orbits to arbitrary sequences in the phase space (Theorems 7, 8). This highlights the importance of the notion of proximality.

In Section 7, we study the "fastness" involved in proximality. Among a few other things, we show that (Corollary 5) if f is an interval map having a periodic point which is not a power of 2, then for every function $\phi : \mathbb{N} \to \mathbb{N}$, there is an uncountable scrambled set $S \subset [0,1]$ for f with the extra property that

$$\liminf_{n\to\infty} \phi(n)|f^n(x) - f^n(y)| = 0 \text{ for every } x, y \in S. \tag{8}$$

To conclude this section, we remark that recently there has been some interesting research happening (see for example, [6, 19]) concerning the rate of recurrence formulated in terms of the first return times to neighborhoods; this may be compared with our work, even though our perspective is slightly different.

2 Preliminaries

Throughout the paper, by a **topological dynamical system** we mean a pair (X, f) where X is a metric space having the property that the Baire Category Theorem holds on every nonempty closed subspace of X in the induced topology (for example, X can be complete or locally compact), and $f : X \to X$ is a continuous map. If $x \in X$, then the **f-orbit** of x is $\{x, f(x), f^2(x), f^3(x), \ldots\}$. If $f^n(x) = x$ for some $n \in \mathbb{N}$, we say that x is a **periodic point** for f. The set of periodic points of f will be denoted by $P(f)$. For $x \in X$ and $U, V \subset X$ write

$$N_f(x, U) = \{n \in \mathbb{N} : f^n(x) \in U\}, \tag{9}$$

$$N_f(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}. \tag{10}$$

A topological dynamical system (X, f) is a **minimal system** (or f is a **minimal map**) if the f-orbit of every $x \in X$ is dense in X. It is easy to see that (X, f) is a minimal system iff X has no proper, nonempty, closed f-invariant subset. An element $x \in X$ is called a **minimal point** if the restriction of f to the orbit-closure of x is minimal. Note that every periodic point is a minimal point, and that every minimal point is a recurrent point. If X is locally compact and if $x \in X$ is a minimal point, it is known [17] that $N_f(x, U)$ is a **syndetic set** (i.e., an infinite set with bounded gaps) for every neighborhood U of x.

We say f is **transitive** if $N_f(U,V) \neq \emptyset$ for any two nonempty open sets $U, V \subset X$, f is **weak mixing** if $f \times f : X^2 \to X^2$ is transitive, and f is **mixing** if $N_f(U,V)$ is cofinite in $\mathbb{N}$ for any two nonempty open sets $U, V \subset X$. We remark that when f is weak mixing, any finite product $f \times \cdots \times f$ is transitive (c.f. [9]).

A pair $(x,y) \in X^2$ is called a **proximal pair** for f if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, and an **asymptotic pair** for f if $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$. A subset $S \subset X$ having at least two points is known as a **scrambled set** for f if for any two distinct $x, y \in S$, (x,y) is a proximal but not an asymptotic pair for f.

Let (X,f), (Y,g) be two topological dynamical systems. We say (X,f) is an **almost one-one extension** of (Y,g) if there is a continuous surjection $h : X \to Y$ such that $h \circ f = g \circ h$ and $\{x \in X : h^{-1}(h(x)) = \{x\}\}$ is residual in X.

In [11], Glasner and King proved a *zero-one law* for transitive homeomorphisms, which is not quite sufficient for us since we need to consider non-invertible continuous maps also. The required version of the law is proved below:

Theorem 1 (Zero-one law)**.** *Let (X,f) be a topological dynamical system. If f is transitive and semi-open, then for any Borel set $A \subset X$ with $f(A) \subset A$, either A or $X \setminus A$ is residual in X.*

Proof. First note that if $D \subset X$ is a nowhere dense closed set, then so is $f^{-1}(D)$ by the semi-openness f. Hence, the f-preimages of first category sets are also of first category. Now, consider a Borel set $A \subset X$ with $f(A) \subset A$. Let $B = X \setminus A$. Since B is Borel, there exists an open set $U \subset X$ such that the symmetric difference $U \Delta B$ is of first category. It suffices to show that either A is of first category or $U = \emptyset$. Let $U_1 = \bigcup_{n=0}^{\infty} f^{-n}(U \cap B)$ and $U_2 = \bigcup_{n=0}^{\infty} f^{-n}(U \setminus B)$. We have $U_1 \subset B$ since $f^{-1}(B) \subset B$, and U_2 is of first category since $U \setminus B$ is of first category. Hence $A \cap (U_1 \cup U_2)$ is of first category. On the other hand, $U_1 \cup U_2 = \bigcup_{n=0}^{\infty} f^{-n}(U)$, which must be a dense open set by the transitivity of f if $U \neq \emptyset$. Thus either A is of first category or $U = \emptyset$. $\square$

Finally, for a subset $M \subset \mathbb{N}$, define the **lower density** of M as

$$\underline{\rho}(M) := \liminf_{n\to\infty} \frac{|M \cap \{1, 2, \ldots, n\}|}{n}. \tag{11}$$

Note that if M is syndetic, then $\underline{\rho}(M) > 0$.

3 Recurrence is fast when there are many periodic points

Let (X, f) be a topological dynamical system, $\phi : \mathbb{N} \to \mathbb{N}$ be a function, and let $R(f, \phi)$ be as defined in the Introduction. Note that $R(f, \phi)$ is a G_δ since

$$R(f, \phi) = \bigcap_{k} \bigcup_{n \geq k} \{x \in X : \phi(n) d(x, f^n(x)) < 1/k\}. \tag{12}$$

Hence, whenever we need to prove the residuality of $R(f, \phi)$, it will suffice to show that $R(f, \phi)$ is dense. Below, we discover that the abundance of periodic points can make all $R(f, \phi)$ "big" and vice versa:

Theorem 2. *Let (X, f) be a topological dynamical system. Then,*
(i) $P(f) = \bigcap_{\phi} R(f, \phi)$. In particular, when $\overline{P(f)} = X$, we get that $R(f, \phi)$ is a dense G_δ in X for every $\phi : \mathbb{N} \to \mathbb{N}$.
(ii) If $K \subset X$ is compact and if $R(f, \phi) \cap K \neq \emptyset$ for every $\phi : \mathbb{N} \to \mathbb{N}$, then $P(f) \cap K \neq \emptyset$. In particular, if X is locally compact, and if $R(f, \phi)$ is a dense G_δ in X for every $\phi : \mathbb{N} \to \mathbb{N}$, then $\overline{P(f)} = X$.
(iii) If X is a locally compact, separable space, and $P(f) = \emptyset$, there exists $\phi : \mathbb{N} \to \mathbb{N}$ with $R(f, \phi) = \emptyset$.

Proof. (i) Clearly, $P(f) \subset R(f, \phi)$ for every $\phi : \mathbb{N} \to \mathbb{N}$. Now, if $x \notin P(f)$ then choose $\phi : \mathbb{N} \to \mathbb{N}$ with $\phi(n) d(x, f^n(x)) \geq 1$ for every n, and we see $x \notin R(f, \phi)$.
(ii) Let if possible $P(f) \cap K = \emptyset$. For each $n \in \mathbb{N}$, the function $x \mapsto d(x, f^n(x))$ from K to $[0, \infty)$ is continuous, and bounded away from 0 by the compactness of K since $K \cap P(f) = \emptyset$. Hence we can define $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(n) d(x, f^n(x)) \geq 1$ for every $x \in K$ and for every $n \in \mathbb{N}$. And then, $R(f, \phi) \cap K = \emptyset$.
(iii) Write $X = \bigcup_{j=1}^{\infty} K_j$ where K_j's are compact. Choose $\phi_j : \mathbb{N} \to \mathbb{N}$ with $R(f, \phi_j) \cap K_j = \emptyset$. Define $\phi : \mathbb{N} \to \mathbb{N}$ as $\phi(n) = \sum_{j=1}^{n} \phi_j(n)$. Then for every j, $R(f, \phi) \subset R(f, \phi_j)$ so that $R(f, \phi) \cap K_j = \emptyset$. □

Corollary 1. *Let $f : [0, 1] \to [0, 1]$ be continuous. Then, the set $R(f)$ of recurrent points is dense in $[0, 1]$ iff $R(f, \phi)$ is a dense G_δ in $[0, 1]$ for every $\phi : \mathbb{N} \to \mathbb{N}$.*

Proof. By [8], for any interval map f, we have $\overline{P(f)} = \overline{R(f)}$. □

Consider a topological dynamical system (X, f). Observe that if f is Lipschitz continuous, and $\phi : \mathbb{N} \to \mathbb{N}$ satisfies $\limsup_{n \to \infty} \phi(n+1)/\phi(n) < \infty$, then $R(f, \phi)$ is f-invariant. Therefore, we can say that $R(f, \phi)$ is either residual or nowhere dense (since it is a G_δ) when f is transitive and semi-open, by the zero-one law. When f is minimal (note: minimal maps are semi-open), we can say a little more:

Proposition 1. *Let (X, f) be a minimal system with f Lipschitz. Then we have that $R(f, \phi)$ is either empty or residual for any $\phi : \mathbb{N} \to \mathbb{N}$ with* $\limsup_{n\to\infty} \phi(n+1)/\phi(n) < \infty$.

Let $\alpha \in [0, 1)$ and let $f_\alpha : S^1 \to S^1$ be the rotation given by $f_\alpha(x) = e^{2\pi i\alpha}x$. Since $d(x, f_\alpha^n(x)) = d(1, f_\alpha^n(1))$ for every $x \in S^1$ and $n \in \mathbb{N}$, we see that for each $\phi : \mathbb{N} \to \mathbb{N}$, $R(f_\alpha, \phi)$ is either empty or the whole of S^1. If α is rational, clearly $R(f_\alpha, \phi) = S^1$, as every point in S^1 is periodic. If α is irrational, we know from Theorem 2(iii) that $R(f_\alpha, \phi) = \emptyset$ for some $\phi : \mathbb{N} \to \mathbb{N}$. We leave it as an open question to determine all ϕ with $R(f_\alpha, \phi) = \emptyset$ for a given irrational α. However, we could prove the following complementary result:

Proposition 2. *For any $\phi : \mathbb{N} \to \mathbb{N}$, the set $\{\alpha \in [0, 1) : R(f_\alpha, \phi) = S^1\}$ is residual in $[0, 1)$.*

Proof. Fix ϕ. Let $Y_n = \{\alpha \in [0, 1) : \phi(n)d(1, f_\alpha^n(1)) < 1/n\}$. It suffices to establish that $\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty Y_n$ is a dense G_δ in $[0, 1)$. Now, Y_n is clearly open. If $k \in \mathbb{N}$, and $J \subset [0, 1)$ is a relatively open nonempty interval, then choose a rational number $p/q \in J$ and note that $f_{p/q}^{kq}(1) = 1$. Thus $\bigcup_{n=k}^\infty Y_n$ is dense in $[0, 1)$. By Baire category theorem, we are through. □

4 A sense in which the recurrence is generically slow

Next, we show that in another sense, the recurrence can be slow for most of the recurrent points. Let (X, f) be a topological dynamical system, let $\phi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, and let $R_L(f, \phi)$ be as defined in the Introduction. Observe that

$$R_L(f, \phi, 1/k) = \bigcup_{r=1}^\infty \bigcup_{s=1}^\infty \{x \in X : \frac{|\{1 \le j \le \phi(n) : d(x, f^j(x)) \le 1/k\}|}{n} \ge 1/r \text{ for every } n \ge s\}, \tag{13}$$

which is an F_σ set, and hence $R_L(f, \phi) = \bigcap_{k=1}^\infty R_L(f, \phi, 1/k)$ is $F_{\sigma\delta}$. Since $R_L(f, \phi)$ is f-invariant, it is either residual or of first category when f is transitive and semi-open, by the zero-one law.

Now, $R_L(f, I_\mathbb{N})$ is the collection of all recurrent points whose set of return times to any of its neighborhood has positive lower density. Moreover, $R_L(f, \phi) \subseteq R_L(f, \psi)$ if $\phi \le \psi$. It is a well-known consequence of Birkhoff's ergodic theorem (c.f. [7]) that if f admits an (ergodic) invariant probability measure, then $R_L(f, I_\mathbb{N})$ (and hence every $R_L(f, \phi)$) has full measure; see also Lemma 1 in the next section. In contrast we show that these sets are often topologically "small".

We find the following obstruction for $R_L(f, \phi)$ to be residual:

Theorem 3. *Let (X, f) be a topological dynamical system with f uniformly continuous. Suppose there exists a proper closed f-invariant set $A \subset X$ with $\overline{\bigcup_{n=0}^{\infty} f^{-n}(A)} = X$. Then for every strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$, we have that $R_L(f, \phi) \cap (X \setminus A)$ is of first category in X, and thus $R_L(f, \phi)$ is not residual in X. If further, f is transitive, then $R_L(f, \phi)$ is of first category in X.*

Proof. Let if possible $\phi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function so that $R_L(f, \phi) \cap (X \setminus A)$ is of second category in X, and we will derive a contradiction. Choose a nonempty open set $U \subset X$ and $\epsilon > 0$ such that $d(A, U) > 2\epsilon$ and $R_L(f, \phi) \cap U$ is of second category in X. Since $R_L(f, \phi) \subset R_L(f, \phi, \epsilon)$ and $R_L(f, \phi, \epsilon)$ is F_σ, we conclude that $R_L(f, \phi, \epsilon) \cap U$ has nonempty interior in X. Now, see that $R_L(f, \phi, \epsilon) \cap U = \bigcup_{r,s \in \mathbb{N}} S(r, s)$, where

$$S(r, s) = \{x \in U : \frac{|\{1 \leq j \leq \phi(n) : d(x, f^j(x)) \leq \epsilon\}|}{n} \geq 1/r \text{ for every } n \geq s\}. \tag{14}$$

By Baire category theorem, $S(r, s)$ is of second category for some $r, s \in \mathbb{N}$. Now, using the f-invariance of A and the uniform continuity of f^n's, choose $\delta_n > 0$ so that $d(z, A) < \delta_n$ implies $d(f^i(z), A) < \epsilon$ for $1 \leq i \leq \phi(2nr)$, and put $V = \bigcup_{n \geq s} f^{-n}(\{z \in X : d(z, A) < \delta_n\})$. Then, V is open. It follows from the hypothesis on A that $\overline{\bigcup_{n=s}^{\infty} f^{-n}(A)} = X$, and hence V is dense in X. Since a dense open set must intersect a set of second category, there exists $x \in V \cap S(r, s)$. Then, $x \in U$ and $d(f^n(x), A) < \delta_n$ for some $n \geq s$. From our choice of ϵ and δ_n, we see that

$$\frac{|\{1 \leq j \leq \phi(2nr) : d(x, f^j(x)) \leq \epsilon\}|}{2nr} \leq \frac{n}{2nr} < \frac{1}{r}. \tag{15}$$

Since $n \geq s$, this implies $x \notin S(r, s)$, a contradiction. This completes the proof of the main statement of the Theorem. The last statement of the Theorem follows by the fact that A must be nowhere dense when f is transitive. □

See [16] for the description of a *subshift of finite type* and its basic properties.

Corollary 2. *Let (X, f) be either a transitive dynamical system on* $[0, 1]$*, or an infinite, transitive, one-sided subshift of finite type. Then $R_L(f, \phi)$ is of first category in X for any strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$.*

Proof. In both the cases, one can find a periodic point $x \in X$ with a dense backward orbit (see [5, 16]). Take A to be the periodic f-orbit of x, and apply Theorem 3. □

Theorem 4. *Let $f : [0, 1] \to [0, 1]$ be a continuous map with $R(f)$ dense. If $f^2 \neq Id$ on any subinterval of positive length, then $R_L(f, \phi)$ is of first category in* $[0, 1]$ *for any strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$.*

Proof. Given the hypothesis, by a decomposition result of Barge and Martin (c.f. [5, 18]), there exist at most countably many closed intervals $J_n \subset [0,1]$ such that

(i) $[0,1] \setminus \bigcup_n J_n$ is nowhere dense,

(ii) Each J_n is f^2-invariant and $f^2|_{J_n}$ is transitive.

From this information and by the above Corollary, it is not difficult to deduce that $R_L(f^2, \phi)$ is of first category in $[0,1]$ for any strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$. Now, it is a small exercise to check that $R_L(f, \phi) \subset R_L(f^2, \phi)$ (in fact, they are equal). □

5 A contrast between measure and topological dynamical systems

In this section we consider the rate of recurrence in terms of certain sets rather than points. We will find that the recurrence is fast for measure dynamical systems, but the recurrence can be arbitrarily slow for topological dynamical systems, even if the system is mixing. The following Lemma should be known:

Lemma 1. *Let (X, T, μ) be a probability measure dynamical system and let $A \subset X$ be with $\mu[A] > 0$. Then $N_T(a, A)$ has positive lower density for μ-almost every $a \in A$.*

Proof. Let $A' = \{a \in A : \underline{\rho}(N_T(a, A)) > 0\}$. Note that A' is measurable. If $\mu[A \setminus A'] > 0$, then by the ergodic decomposition of measures (c.f. [9]), there is a T-invariant ergodic probability measure ν such that $\nu[A \setminus A'] > 0$. Then, by Birkhoff's ergodic theorem (c.f. [7]), for ν-almost every $a \in A \setminus A'$, and hence for some $a \in A \setminus A'$, we have $\underline{\rho}(N_T(a, A \setminus A')) = \nu[A \setminus A'] > 0$. Thus $\underline{\rho}(N_T(a, A)) > 0$, and therefore $a \in A'$, a contradiction. □

Kindly recall the functions $\beta_T(A, n)$, $\beta_f(U, n)$ defined in the Introduction. Note that the growth rate of these functions with respect to n is a measure of the slowness of recurrence. We show below that the growth rate of $\beta_T(A, n)$ is linear, where as the growth rate of $\beta_f(U, n)$ can be arbitrarily high.

Theorem 5. *Let (X, T, μ) be a probability measure dynamical system and let $A \subset X$ be with $\mu[A] > 0$. Then $\limsup_{n\to\infty} \dfrac{\beta_T(A, n)}{n} < \infty$.*

Proof. Let $A' = \{a \in A : \underline{\rho}(N_T(a, A)) > 0\}$. By the Lemma, $\mu[A'] = \mu[A] > 0$. Note that $A' = \bigcup_{r,s\in\mathbb{N}} A(r, s)$, where

$$A(r, s) = \{a \in A : \frac{|\{1 \le j \le n : T^j a \in A\}|}{n} \ge 1/r \text{ for every } n \ge s\}. \tag{16}$$

So there exist r, s with $\mu[A(r,s)] > 0$. Since $A(r,s) \subset A(r+1,s)$, we may assume $r \geq s$. As the number of n-element subsets of $\{1, 2, \ldots, nr\}$ is finite, it follows that for every $n \in \mathbb{N}$, there are natural numbers $j_1 < \cdots < j_n \leq nr$ such that $\mu[A \cap T^{-j_1}A \cap \cdots \cap T^{-j_n}A] > 0$. Thus $\beta_T(A,n) \leq nr$ for every $n \in \mathbb{N}$. □

As an aside, observe that Theorem 5 supplies a necessary condition for a topological dynamical system to admit an invariant Borel probability measure of full support.

Theorem 6. *Let $\phi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Then there exists a topological dynamical system (X, f) with X compact, and a nonempty open set $U \subset X$ such that*
(i) f is mixing.
(ii) $\beta_f(U,n) > \phi(n)$ for every $n \in \mathbb{N}$.

Proof. If w is a word over $\{0,1\}$, let $|w|$ denote its length and let $\alpha(w,1)$ denote the number of 1's in w. For example, if $w = 10110$, then $|w| = 5$ and $\alpha(w,1) = 3$. Let X be the collection of all $x \in \{0,1\}^{\mathbb{Z}}$ such that every word w appearing in x satisfies the following:

(i) If $|w| \leq \phi(1)$, then $\alpha(w,1) \leq 1$.

(ii) If $\phi(n) < |w| \leq \phi(n+1)$, then $\alpha(w,1) \leq n$.

Then, X is nonempty ($\because 0^\infty \in X$), closed in $\{0,1\}^{\mathbb{Z}}$, and invariant under the shift map $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$, so that (X, σ) is a compact topological dynamical system.

Claim-1: (X, σ) is mixing.

For $x \in \{0,1\}^{\mathbb{Z}}$ and integers $i \leq j$, let us write $x_{[i,j]}$ for the word $x_i x_{i+1} \cdots x_j$. Now, let $x, y \in X$ and $n \in \mathbb{N}$ be given. Let $w = x_{[-n,n]}$ and $v = y_{[-n,n]}$. Then it is easy to see that there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, the element $z(k) \in \{0,1\}^{\mathbb{Z}}$ defined as $z(k) = 0^\infty w 0^k v 0^\infty$ belongs to X. We have $z(k)_{[-n,n]} = x_{[-n,n]}$ and $[\sigma^{2n+1+k}(z(k))]_{[-n,n]} = y_{[-n,n]}$ for every $k \geq k_0$. Since $x, y \in X$ and $n \in \mathbb{N}$ were arbitrary, Claim-1 follows.

Let $U = \{x \in X : x_0 = 1\}$. Then U is open, and nonempty ($\because 0^\infty 1 0^\infty \in U$).

Claim-2: $\beta_\sigma(U,n) > \phi(n)$ for every $n \in \mathbb{N}$.

Let $j_1 < \cdots < j_n$ be natural numbers such that $U \cap \sigma^{-j_1}(U) \cap \cdots \cap \sigma^{-j_n}(U) \neq \emptyset$. If y is an element of this intersection, then $\alpha(y_{[0,j_n]}, 1) \geq n+1$ by the definition of U. This implies by the definition of X that $j_n + 1 > \phi(n+1)$, and therefore $j_n \geq \phi(n+1) > \phi(n)$. Since $y \in U \cap \sigma^{-j_1}(U) \cap \cdots \cap \sigma^{-j_n}(U)$ was arbitrary, this proves Claim-2. □

Remark 1. If $\phi : \mathbb{N} \to \mathbb{N}$ satisfies $\limsup_{n\to\infty}(\phi(n+1) - \phi(n)) = \infty$, then in the above example it may be checked that 0^∞ is the only minimal point in (X, σ). On the other hand, it is easy to prove that if (X, f) is a locally compact topological dynamical system with a dense set of minimal points, then $\limsup_{n\to\infty} \dfrac{\beta_f(U,n)}{n} < \infty$ for every nonempty open $U \subset X$ (using the fact that every syndetical subset of $\mathbb{N}$ has positive lower density).

6 Characterizing weak mixing and mixing

The purpose of this section is to highlight the importance of the notion of proximality by providing characterizations of weak mixing and mixing in terms of the proximality of orbits to arbitrary sequences in the phase space.

Theorem 7. *Let (X, f) be a topological dynamical system where X is locally compact. Then the following are equivalent:*
(i) f is weak mixing.
(ii) For every sequence (x_n) in X with $\overline{\{x_n : n \in \mathbb{N}\}}$ compact, there exists a dense G_δ set $Y \subset X$ such that $\liminf_{n\to\infty} d(x_n, f^n(y)) = 0$ for every $y \in Y$.
(iii) For any two non-empty open sets $U, V \subset X$ with $\overline{V}$ compact, and any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $f^n(U)$ intersects every open ball of radius $\geq \epsilon$ contained in V.

Proof. $(i) \Rightarrow (ii)$: Given $x_n \in X$ with $\overline{\{x_n : n \in \mathbb{N}\}}$ compact, and $k \in \mathbb{N}$, let $Y_k = \{y \in X : d(x_n, f^n(y)) < 1/k$ for some $n \geq k\}$. Clearly Y_k is open. In view of Baire category theorem, it remains to show that Y_k is dense, for then we may take $Y = \bigcap_{k=1}^{\infty} Y_k$. Let $U \subset X$ be nonempty, open. Cover $\{x_n : n \in \mathbb{N}\}$ with finitely many balls $B_1, \ldots, B_p$, each of diameter $< 1/k$. Since f is weak mixing, we can find $n \in \bigcap_{i=1}^{p} N_f(U, B_i)$ with $n \geq k$. Choose $i \in \{1, \ldots, p\}$ such that $x_n \in B_i$ and let $y \in U$ be such that $f^n(y) \in B_i$. Then, $d(x_n, f^n(y)) \leq diam[B_i] < 1/k$. Thus $y \in U \cap Y_k$.
$(ii) \Rightarrow (iii)$: If (iii) is false, then there exist nonempty open sets $U, V \subset X$ with $\overline{V}$ compact, and $\epsilon > 0$ such that for each $n \in \mathbb{N}$, there is an open ball B_n of radius $\geq \epsilon$ satisfying $B_n \subset V$ and $f^n(U) \cap B_n = \emptyset$. Let x_n be the center of B_n. Then $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact and $d(x_n, f^n(y)) \geq \epsilon$ for every $y \in U$ and every $n \in \mathbb{N}$. So (ii) cannot hold.
$(iii) \Rightarrow (i)$: Clearly (iii) implies f is transitive. Now let $U_1, U_2, V_1, V_2 \subset X$ be nonempty open sets. Choose $k \in \mathbb{N}$ such that $U := U_1 \cap f^{-k}(U_2)$ is nonempty. Let $V_3 \subset V_1$, $V_4 \subset f^{-k}(V_2)$ be nonempty open sets with compact closures. Put $V = V_3 \cup V_4$. Let $\epsilon > 0$ be so that each of V_3, V_4 contains a ball of radius $\geq \epsilon$. Applying (iii) to the pair U, V and this ϵ, we find $n \in \mathbb{N}$ such that $f^n(U) \cap V_i \neq \emptyset$ for $i = 3, 4$. Hence $f^n(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$. □

Corollary 3. *Let X be a compact metric space and let $\mathcal{F}$ be a countable collection of weak mixing maps on X. Then, for every $x \in X$ there exists a dense G_δ set $Y \subset X$ such that* $\liminf_{n\to\infty} d(f^n(x), g^n(y)) = 0$ *for every $f, g \in \mathcal{F}$ and every $y \in Y$.*

Theorem 8. *Let (X, f) be a topological dynamical system. Then the following are equivalent.*
(i) f is mixing.
(ii) For every sequence (x_n) in X and for any subsequence (x_{n_k}) with $\overline{\{x_{n_k} : k \in \mathbb{N}\}}$ compact, there exists a dense G_δ set $Y \subset X$ such that $\liminf_{k\to\infty} d(x_{n_k}, f^{n_k}(y)) = 0$ *for every $y \in Y$.*

Proof. The proof of $(i) \Rightarrow (ii)$ is similar to the proof of $(i) \Rightarrow (ii)$ in Theorem 7, and is left to the reader.
$(ii) \Rightarrow (i)$: If f is not mixing, there exist nonempty open sets $U, V \subset X$ and an increasing sequence (n_k) of natural numbers such that $f^{n_k}(U) \cap V = \emptyset$ for every k. Let (x_n) be a constant sequence in V. Then $\liminf_{k\to\infty} d(x_{n_k}, f^{n_k}(y)) > 0$ for every $y \in U$. So (ii) cannot be true. □

Remark 2. A corollary similar to that of Corollary 3 can be formulated. Moreover, note that our results improve some findings of [12].

7 The rate of proximality

Now, we carry out a study about proximal pairs with an approach similar to the one applied for recurrent points in Section 3. We will find similarities. In fact we could have attempted a unified treatment, which was avoided mainly due to the fear of an overcrowding of highly abstract notations. Let (X, f) be a topological dynamical system, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a function. Define

$$Prox(f, \phi) = \{(x, y) \in X^2 : \liminf_{n\to\infty} \phi(n)d(f^n(x), f^n(y)) = 0\}. \tag{17}$$

Note that $Prox(f, \phi)$ is a G_δ subset of X^2 since

$$Prox(f, \phi) = \bigcap_k \bigcup_{n\geq k} \{(x, y) \in X^2 : \phi(n)d(f^n(x), f^n(y)) < 1/k\}. \tag{18}$$

Below we see that the injectivity of f can be an obstruction to $Prox(f, \phi)$ being "big".

Theorem 9. *Let (X, f) be a topological dynamical system. Then,*
(i) $\bigcup_{n=1}^{\infty} (f \times f)^{-n}(\Delta_X) = \bigcap_\phi Prox(f, \phi)$, where Δ_X is the diagonal of X^2.
(ii) If $K \subset X^2$ is compact and if $Prox(f, \phi) \cap K \neq \emptyset$ for every $\phi : \mathbb{N} \to \mathbb{N}$, then $\bigcup_{n=1}^{\infty} (f \times f)^{-n}(\Delta_X)$ intersects K. In particular if X is locally compact, and if $Prox(f, \phi)$ is a dense G_δ in X^2 for every $\phi : \mathbb{N} \to \mathbb{N}$, then $\overline{\bigcup_{n=1}^{\infty} (f \times f)^{-n}(\Delta_X)} = X^2$.

(iii) If X is a locally compact separable space, and if f is injective, then there exists $\phi : \mathbb{N} \to \mathbb{N}$ such that $Prox(f, \phi) = \Delta_X$.

Proof. (i) The inclusion $\bigcup_{n=1}^{\infty}(f \times f)^{-n}(\Delta_X) \subset \bigcap_{\phi} Prox(f, \phi)$ is obvious. Now, if $(x, y) \in X^2$ is such that $f^n(x) \neq f^n(y)$ for every $n \in \mathbb{N}$, we can choose $\phi : \mathbb{N} \to \mathbb{N}$ with $\phi(n)d(f^n(x), f^n(y)) \geq 1$ for every $n \in \mathbb{N}$, and then $(x, y) \notin Prox(f, \phi)$.
(ii) This is similar to the proof of Theorem 2(ii).
(iii) Can be deduced from (ii) as follows. Write $X^2 \setminus \Delta_X = \bigcup_{j=1}^{\infty} K_j$ where K_j's are compact. Choose $\phi_j : \mathbb{N} \to \mathbb{N}$ with $Prox(f, \phi) \cap K_j = \emptyset$ and take $\phi : \mathbb{N} \to \mathbb{N}$ as $\phi(n) = \sum_{j=1}^{n} \phi_j(n)$. □

Corollary 4. *Let (X, f) denote either a dynamical system on $[0, 1]$ or an infinite, one-sided subshift of finite type. If f is mixing, then $Prox(f, \phi)$ is a dense G_δ in X^2 for every $\phi : \mathbb{N} \to \mathbb{N}$.*

Proof. The systems mentioned have the property that $\bigcup_{n=1}^{\infty}(f \times f)^{-n}(\Delta_X)$ is dense in X^2. □

Remark 3. Let (X, f) be either a dynamical system on the interval or a subshift of finite type. Then it is known that f is mixing if f^n is transitive for every $n \in \mathbb{N}$.

Question: Let (X, f) be a two-sided mixing subshift of finite type. Since f is injective, by Theorem 9(iii), $Prox(f, \phi) = \Delta_X$ for some $\phi : \mathbb{N} \to \mathbb{N}$. On the other hand, $Prox(f, I_{\mathbb{N}})$ is residual in X^2 since f is mixing. Is it possible to determine all $\phi : \mathbb{N} \to \mathbb{N}$ such that $Prox(f, \phi)$ is residual?

Next, we establish that when all $Prox(f, \phi)$'s are "big", all $R_L(f \times f, \psi)$'s have to be "small".

Proposition 3. *Let (X, f) be a topological dynamical system, where X has no isolated points and f is uniformly continuous. Assume that $Prox(f, \phi)$ is residual in X^2 for every $\phi : \mathbb{N} \to \mathbb{N}$. Then $R_L(f \times f, \psi)$ is of first category in X^2 for every strictly increasing $\psi : \mathbb{N} \to \mathbb{N}$.*

Proof. Suppose on the contrary there is a strictly increasing $\psi : \mathbb{N} \to \mathbb{N}$ such that $R_L(f \times f, \psi)$ is of second category in X^2. We may choose nonempty open sets $U, V \subset X$ and $\epsilon > 0$ such that

(i) $d(U, V) > 2\epsilon$,

(ii) $R_L(f \times f, \psi) \cap (U \times V)$ is of second category in X^2.

Since $R_L(f \times f, \psi) \subset R_L(f \times f, \psi, \epsilon)$ and $R_L(f \times f, \psi, \epsilon)$ is F_σ, we get that $R_L(f \times f, \psi, \epsilon) \cap (U \times V)$ has nonempty interior. Now, the proof is similar to that of Theorem 3. Note that $R_L(f \times f, \psi, \epsilon) \cap (U \times V) = \bigcup_{r,s \in \mathbb{N}} S(r, s)$, where

$$S(r, s) = \{(x, y) \in U \times V : \frac{|\{0 < j \leq \psi(n) : d(x, f^j(x)) + d(y, f^j(y)) \leq \epsilon\}|}{n} \geq 1/r \text{ for every } n \geq s\}. \tag{19}$$

Hence, there exist $r, s \in \mathbb{N}$ such that $S(r,s)$ is of second category in X^2. Now, using the uniform continuity of f^n's define $\phi : \mathbb{N} \to \mathbb{N}$ such that $d(a,b) < 1/\phi(n)$ implies $d(f^i(a), f^i(b)) < \epsilon$ for $1 \le i \le \psi(2nr)$.

Claim: $Prox(f, \phi)$ is not residual in X^2.

Otherwise, choose $(x,y) \in Prox(f,\phi) \cap S(r,s)$. Since $(x,y) \in U \times V$, $d(x,y) > 2\epsilon$. Since $(x,y) \in Prox(f,\phi)$, there exists $n \ge s$ such that $\phi(n) d(f^n(x), f^n(y)) < 1$. Then, $d(f^{n+i}(x), f^{n+i}(y)) < \epsilon$ for $1 \le i \le \psi(2nr)$. From our choice of ϵ, and by triangle inequality, we see that

$$\frac{|\{1 \le j \le \psi(2nr) : d(x, f^j(x)) + d(y, f^j(y)) \le \epsilon\}|}{2nr} \le \frac{n}{2nr} < \frac{1}{r}. \tag{20}$$

Since $n \ge s$, this implies $(x,y) \notin S(r,s)$, a contradiction. This proves the Claim, and completes the proof. □

Question: Let (X, f) be a topological dynamical system, and $\phi : \mathbb{N} \to \mathbb{N}$ be strictly increasing. If $R_L(f, \phi)$ is residual (second category) in X, can we say that $R_L(f \times f, \phi)$ is residual (second category) in X^2?

The concept of a scrambled set was first introduced by Li and Yorke [15] in order to formulate the idea of *chaos*, and there has been a lot of work on this concept, see for instance [3, 4].

Theorem 10. *Let (X, f) be a topological dynamical system, and suppose there exist $k, m \in \mathbb{N}$ and a closed f^k-invariant set $Y \subset X$ such that (Y, f^k) is an almost one-one extension of the one-sided shift $(\sum_m, \sigma)$ on m symbols ($m \ge 2$). Then for every $\phi : \mathbb{N} \to \mathbb{N}$, there is an uncountable scrambled set $S \subset X$ for f satisfying the extra property that $\liminf_{n\to\infty} \phi(n) d(f^n(x), f^n(y)) = 0$ for all $x, y \in S$.*

Proof. Note that we are done if we prove two things:

(i) For any $y \in Y$, the set $\{z \in Y : \lim_{n\to\infty} d(f^n(y), f^n(z)) = 0\}$ is of first category in Y.

(ii) $Prox(f, \phi) \cap Y^2$ is residual in Y^2.

Since $(\sum_m, \sigma)$ is mixing, and mixing is preserved by almost one-one extensions [1], (Y, f^k) is mixing. Hence by [13], for any $y \in Y$, the set $\{z \in Y : \lim_{n\to\infty} d(f^{kn}(y), f^{kn}(z)) = 0\}$ is of first category in Y. Now, (i) follows easily. Again, through the almost one-one extension, (Y, f^k) gets the property that $\bigcup_{n=1}^{\infty} (f^k \times f^k)^{-n}(\Delta_Y)$ is dense in Y^2. Hence $Prox(f^k|_Y, \psi)$ is residual in Y^2 for every $\psi : \mathbb{N} \to \mathbb{N}$. In particular, it holds for ψ defined as $\psi(n) = \phi(kn)$. And then, $Prox(f^k|_Y, \psi) \subset Prox(f, \phi) \cap Y^2$, which proves (ii). □

Corollary 5. *Let* $f : [0,1] \to [0,1]$ *be a continuous map having a periodic point whose period is not a power of 2. Then, for every* $\phi : \mathbb{N} \to \mathbb{N}$*, there is an uncountable scrambled set* $S \subset [0,1]$ *for* f *satisfying the extra property that* $\liminf_{n\to\infty} \phi(n)|f^n(x) - f^n(y)| = 0$ *for all* $x, y \in S$*.*

Proof. By known results (Theorem 14 and the proof of Proposition 15 in Chapter 2 of [5], or Theorem 4.3.10 and Proposition 5.2.4 of [18]), f satisfies the hypothesis of Theorem 10. □

Example: For the parameter range $1 + \sqrt{8} < r \leq 4$, the logistic map $f_r : [0,1] \to [0,1]$ given by $f_r(x) = rx(1-x)$ has a periodic point of period 3 (c.f. [14]), and hence the conclusion of the above Corollary holds. For $r = 4$, the logistic map $f_4 : [0,1] \to [0,1]$ is mixing and (hence) has a dense set of recurrent points so that the conclusions of Corollaries 1, 2 and 4 hold for f_4.

We may continue our investigation by defining $Prox_L(f,\phi)$ next. For instance, the patient reader may verify that, by imitating the techniques of Section 4, the following two statements could be proved:

1. Let (X, f) be a topological dynamical system with f uniformly continuous. Suppose there exist $\epsilon > 0$ and a closed $f \times f$-invariant set $A \subset X^2$ at a distance $> 2\epsilon$ from Δ_X such that $\overline{\bigcup_{n=0}^{\infty}(f \times f)^{-n}(A)} = X^2$. Then for every strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$, we have that $Prox_L(f,\phi,\epsilon)$ is of first category in X^2, and hence $Prox_L(f,\phi)$ is of first category in X^2.
2. Let (X, f) be either a transitive dynamical system on $[0,1]$, or an infinite, transitive, one-sided subshift of finite type. Then $Prox_L(f,\phi)$ is of first category in X^2 for any strictly increasing $\phi : \mathbb{N} \to \mathbb{N}$. [Hint: take two disjoint periodic orbits P, Q suitably and put $A = P \times Q$.]

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