Orbits of Darboux-like real functions

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Abstract

We show that, with respect to the dynamics of iteration, Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$ can exhibit some strange properties which are impossible for continuous functions. To be precise, we show that (i) there is an extendable function from $\mathbb{R}$ to $\mathbb{R}$ which is 'universal for orbits' in the sense that it possesses every orbit of every function from $\mathbb{R}$ to $\mathbb{R}$ up to an arbitrary small translation, and which has orbits asymptotic to any real sequence, (ii) there is a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $n \in \mathbb{N}$, f^n is almost continuous and the graph of f^n is dense in $\mathbb{R}^2$, in spite of the fact that all f-orbits are finite. To prove (i) we assume the Continuum Hypothesis.

Key words: Darboux-like function, orbit, topological transitivity, real sequence, Continuum Hypothesis.
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1 Introduction

In the study of the iterative dynamics of functions $f : X \to X$ of a metric space X, two of the basic questions are the following:

(i) Which "types" of orbits can coexist in a system (X, f)?

(ii) If all orbits in a system (X, f) are "simple", is the global dynamics "simple"?

First we make a peripheral investigation regarding these questions in the case of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Of course, (i) is a huge problem and we do not answer it in any non-trivial sense. Using a result from [6] we merely observe that there are uncountably many "different types" of possible orbits such that a continuous function can possess at most one of those "types". We also deduce that the answer to the second question is affirmative in a certain sense for continuous functions $f : \mathbb{R} \to \mathbb{R}$.

After these preliminary observations, we examine the case of *Darboux-like functions*, functions satisfying generalized notions of continuity, $f : \mathbb{R} \to \mathbb{R}$. We remark that the dynamics of *Darboux-like functions* was considered before. For instance, it is known that there exists an *almost continuous* function $f : \mathbb{R} \to \mathbb{R}$ to which *Sarkovski's Theorem* cannot be extended [5]. For some positive results, see [2], [8], [9].

With respect to the two basic questions mentioned above, the results we obtain about *Darboux-like functions* are drastically different from those we get for continuous functions. We show that there is an *extendable function* from $\mathbb{R}$ to $\mathbb{R}$ which is 'universal for orbits' in the sense that it possesses every orbit of every function from $\mathbb{R}$ to $\mathbb{R}$ up to an arbitrary small translation, and which has orbits asymptotic to any real sequence. We also show that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $n \in \mathbb{N}$, f^n is *almost continuous* and the graph of f^n is dense in $\mathbb{R}^2$, in spite of the fact that all f-orbits are finite.

2 Preliminaries

If $f : X \to X$ is a function of a metric space we call the pair (X, f) a **dynamical system**. For $n \in \mathbb{N}$, by f^n we mean the n-fold self-composition of f. For $x \in X$, the f**-orbit** of the point x is $\{x, f(x), f^2(x), f^3(x), \ldots\}$, which we denote by $O_f(x)$. Let $G_f := \{(x, f(x)) : x \in X\}$ denote the graph of f. We say f is **topologically transitive** if $\bigcup_{n=1}^{\infty} G_{f^n}$ is dense in X^2. One may refer to [3] to appreciate the role of topological transitivity in the study of *chaos*. The following fact is well-known, and can easily be deduced using the Baire Category Theorem.

Proposition 1. *Let X be a complete, second countable metric space without isolated points (e.g. $X = \mathbb{R}$). Then for a continuous function $f : X \to X$ the following are equivalent:*
(i) f is topologically transitive .
(ii) there exists a point $x \in X$ whose f-orbit is dense in X.
(iii) $\{x \in X : O_f(x)$ is dense in $X\}$ is a dense G_δ subset of X.

For a metric space X, let $\mathcal{O}_X$ be the collection of all sequences in X which can be realized as orbits of functions $f : X \to X$ (need not be continuous). That is, $\mathcal{O}_X = \{(x_n)_{n=1}^{\infty} \in X^{\mathbb{N}} :$ there is $f : X \to X$ with $f(x_n) = x_{n+1}$ for all $n \in \mathbb{N}\}$. We put an equivalence relation on $\mathcal{O}_X$ by defining $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$ if there is a homeomorphism $h : X \to X$ such that $h(x_n) = y_n$ for every $n \in \mathbb{N}$. Orbits in the same equivalence class are referred to as **orbits of the same type**. The proof of the following is straightforward:

Lemma 1. *Let $f, g : X \to X$ be continuous, let $x, y \in X$ and let $h : X \to X$ be a homeomorphism such that $h(f^n(x)) = g^n(y)$ for $n = 0, 1, 2, \ldots$. If $O_f(x)$ is dense in X, then $h \circ f = g \circ h$.*

For $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in \mathcal{O}_{\mathbb{R}}$ we say $(x_n)_{n=1}^{\infty}$ is a **translate** of $(y_n)_{n=1}^{\infty}$ if there is $b \in \mathbb{R}$ such that $x_n = y_n + b$ for every $n \in \mathbb{N}$. Note that this is stronger than saying $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are of the same type.

A function $f : \mathbb{R} \to \mathbb{R}$ is called **Darboux** if $f(A)$ is connected for every connected subset $A \subset \mathbb{R}$. A classic Theorem of Darboux (c.f. [7]) says that if $g : \mathbb{R} \to \mathbb{R}$ is differentiable, then its derivative is Darboux. In search of a nice characterization of derivatives (which is still unavailable), many properties which are close to the Darboux property have been studied by various authors, see [1], [4] and the references therein. Functions satisfying these generalized continuity properties are collectively known as **Darboux-like functions**.

In this article, we will consider two classes of Darboux-like functions. A function $f : \mathbb{R} \to \mathbb{R}$ is called **almost continuous** if any open subset U of $\mathbb{R}^2$ containing the graph of f contains the graph of some continuous function $g : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an **extendable function** if there exists a function $F : \mathbb{R} \times [0, 1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in \mathbb{R}$ and such that the graph of $F|_Z$ is a connected subset of $\mathbb{R} \times [0, 1] \times \mathbb{R}$ for every connected subset $Z \subset \mathbb{R} \times [0, 1]$. For $f : \mathbb{R} \to \mathbb{R}$, it is known that (c.f. [4])

continuous $\Longrightarrow$ extendable $\Longrightarrow$ almost continuous $\Longrightarrow$ Darboux,

where all the implications are strict. See [1], [4] for more information concerning Darboux-like functions.

As usual, $\mathfrak{c}$ will denote the cardinality of $\mathbb{R}$. A subset $A \subset \mathbb{R}$ is said to be **$\mathfrak{c}$-dense** if the cardinality of $A \cap J$ is $\mathfrak{c}$ for every nondegenerate interval $J \subset \mathbb{R}$.

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3 Continuous $f : \mathbb{R} \to \mathbb{R}$

In the case of a continuous function $f : \mathbb{R} \to \mathbb{R}$, we have the following partial answers regarding the two basic questions mentioned in the beginning of the article.

Proposition 2. *There is an uncountable set $S \subset \mathcal{O}_{\mathbb{R}}$ such that*

(i) each member of S is an orbit of some continuous function from $\mathbb{R}$ to $\mathbb{R}$,

(ii) no two distinct members of S are of the same type,

(iii) if $f:\mathbb{R}\to\mathbb{R}$ is continuous, then at most one member of S is of the same type as that of some orbit of f.

Proof. By [6], there exists an uncountable family $\{f_\alpha : \alpha \in \Lambda\}$ of continuous, topologically transitive functions from $\mathbb{R}$ to $\mathbb{R}$ such that for any homeomorphism $h:\mathbb{R}\to\mathbb{R}$ one has $h\circ f_\alpha \neq f_\beta \circ h$ for every $\alpha,\beta\in\Lambda$ with $\alpha\neq\beta$. Now, by Proposition 1, choose $x_\alpha\in\mathbb{R}$ so that $O_{f_\alpha}(x_\alpha)$ is dense in $\mathbb{R}$. Put $S=\{O_{f_\alpha}(x_\alpha):\alpha\in\Lambda\}$. Then (i) is clearly true, and (ii), (iii) are satisfied because of Lemma 1. □

Proposition 3. *Let X be an infinite complete metric space without isolated points (e.g. $X=\mathbb{R}$). If $f:X\to X$ is a continuous function such that $O_f(x)$ is finite for each $x\in X$, then $\bigcup_{n=1}^{\infty} G_{f^n}$ is nowhere dense in X^2.*

Proof. Let $U,V\subset X$ be nonempty open sets. Our aim is to find nonempty open sets $U'\subset U$ and $V'\subset V$ such that $f^n(U')\cap V'=\emptyset$ for every $n\in\mathbb{N}$. For $m,k\in\mathbb{N}$, let $A(m,k)=\{x\in X : f^{m+k}(x)=f^m(x)\}$. Since f is continuous, each $A(m,k)$ is closed. Also, by hypothesis X is the union of $A(m,k)$'s. Therefore by Baire Category Theorem, there exist $m,k\in\mathbb{N}$ such that $A(m,k)\cap U$ has nonempty interior, say W. Fix $x\in W$ and choose $\epsilon>0$ small enough so that $V\setminus\bigcup_{j=0}^{m+k}B(f^j(x),\epsilon)$ contains an open ball, say V'. Now using continuity, choose $\delta>0$ so that $f^j(B(x,\delta))\subset B(f^j(x),\epsilon)$ for $0\le j\le m+k$. Put $U'=W\cap B(x,\delta)$. □

In the next two sections we show that the behavior of Darboux-like functions is very different in comparison with the last two Propositions.

4 An extendable function $f:\mathbb{R}\to\mathbb{R}$

The following sufficient condition (c.f. [1]) for a function $f:\mathbb{R}\to\mathbb{R}$ to be an extendable function, will be useful.

Proposition 4. *Let $A\subset\mathbb{R}$ be a $\mathfrak{c}$-dense set which is F_σ and of first category. Then there exists an extendable function $\phi:\mathbb{R}\to\mathbb{R}$ with the property that if $f:\mathbb{R}\to\mathbb{R}$ is any function such that $f(a)=\phi(a)$ for every $a\in A$, then f is also an extendable function.*

We show that all types of orbits can coexist in a strong sense for an extendable function. Moreover, the function can be chosen so that it has orbits asymptotic to any real sequence - this may be of interest since in the theory of iterative dynamics, one is concerned with the asymptotic behavior of orbits.

Proposition 5. *Assuming the Continuum Hypothesis, there is an extendable function* $f : \mathbb{R} \to \mathbb{R}$ *with the following properties:*
(i) for any function $g : \mathbb{R} \to \mathbb{R}$*, any* $x \in \mathbb{R}$ *and any* $\epsilon > 0$*, there exist* $y \in \mathbb{R}$ *and* $b \in (0, \epsilon)$ *such that* $f^n(y) = g^n(x) + b$ *for* $n = 0, 1, 2, \ldots$*,*
(ii) for any real sequence $(r_n)_{n=0}^{\infty}$*, and any decreasing sequence* $(\epsilon_n)_{n=0}^{\infty}$ *of positive reals converging to 0, there exists* $s \in \mathbb{R}$ *such that* $|r_n - f^n(s)| < \epsilon_n$ *for* $n = 0, 1, 2, \ldots$*.*

Proof. Let $A \subset \mathbb{R}$ be a $\mathfrak{c}$-dense, F_σ set of first category. Let $\phi : \mathbb{R} \to \mathbb{R}$ be as in Proposition 4. To get the required function $f : \mathbb{R} \to \mathbb{R}$, first we define f on A as $f(a) = \phi(a)$ for $a \in A$. Then by Proposition 4, $f : \mathbb{R} \to \mathbb{R}$ will be an extendable function irrespective of how we define f on $\mathbb{R} \setminus A$. We will define f on $\mathbb{R} \setminus A$ through a process of transfinite induction.

Recall that $\mathcal{O}_\mathbb{R} \subset \mathbb{R}^\mathbb{N}$ is the collection of orbits of all functions from $\mathbb{R} \to \mathbb{R}$. Let $\mathcal{E} \subset \mathbb{R}^\mathbb{N}$ be the collection of all decreasing sequences $(\epsilon_n)_{n=1}^{\infty}$ of positive reals converging to 0. Note that the sets $\mathbb{R}^\mathbb{N}$, $\mathcal{O}_\mathbb{R}$, $\mathcal{E}$ all have cardinality $\mathfrak{c}$. Let $\mathcal{A} = \mathbb{R}^\mathbb{N} \times \mathcal{O}_\mathbb{R} \times \mathcal{E}$. Then, $\mathcal{A}$ also has cardinality $\mathfrak{c}$. Since we assume the Continuum Hypothesis, we can index the elements of $\mathcal{A}$ using the first uncountable ordinal Ω as $\mathcal{A} = \{(R_\alpha, X_\alpha, E_\alpha) : \alpha < \Omega\}$, where we write $R_\alpha = (r_{\alpha,n})_{n=1}^{\infty} \in \mathbb{R}^\mathbb{N}$, $X_\alpha = (x_{\alpha,n})_{n=1}^{\infty} \in \mathcal{O}_\mathbb{R}$ and $E_\alpha = (\epsilon_{\alpha,n})_{n=1}^{\infty} \in \mathcal{E}$.

Let $D_0 = A$, where f is already defined.

Suppose that for some $\alpha < \Omega$ and all $\beta < \alpha$ we have chosen $D_\beta \subset \mathbb{R}$ such that each D_β is of first category and that f is defined on D_β. Note that $\bigcup_{\beta<\alpha} D_\beta$ is also of first category. The α^{th} step is done as follows.

Consider $(R_\alpha, X_\alpha, E_\alpha) \in \mathcal{A}$. Using Baire Category Theorem, inductively choose $a_{\alpha,n} \in (0, \epsilon_{\alpha,n})$ such that all terms $s_{\alpha,n} := r_{\alpha,n} + a_{\alpha,n}$, $n \in \mathbb{N}$, are distinct and such that $s_{\alpha,n}$'s do not belong to the first category set $\bigcup_{\beta<\alpha} D_\beta$. Next, again by the help of Baire Category Theorem, choose a constant $b_\alpha \in (0, \epsilon_{\alpha,1})$ such that the elements $t_{\alpha,n} := x_{\alpha,n} + b_\alpha$, for $n \in \mathbb{N}$, do not belong to the first category set $\{s_{\alpha,n} : n \in \mathbb{N}\} \cup [\bigcup_{\beta<\alpha} D_\beta]$. Define $f(s_{\alpha,n}) = s_{\alpha,n+1}$ (possible since $s_{\alpha,n}$'s are distinct), and $f(t_{\alpha,n}) = t_{\alpha,n+1}$ (possible since $(t_{\alpha,n})_{n=1}^{\infty}$ is a translate of an orbit of some function). Put $D_\alpha = \{s_{\alpha,n} : n \in \mathbb{N}\} \cup \{t_{\alpha,n} : n \in \mathbb{N}\} \cup [\bigcup_{\beta<\alpha} D_\beta]$. To proceed with the transfinite induction, note that D_α is of first category.

Finally, put $f(y) = 0$ for any possible $y \in \mathbb{R} \setminus \bigcup_{\alpha<\Omega} D_\alpha$. It is not difficult to verify that the function f satisfies all the requirements. □

Remark: It was pointed out to the author that in the α^{th} step of the above proof, $s_{\alpha,n}$ can be chosen even if the complement of $\bigcup_{\beta<\alpha} D_\beta$ is only $\mathfrak{c}$-dense. Therefore, by some extra work involving the consideration of a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, etc., part (ii) might be proved without assuming the Continuum Hypothesis. However, the author does not know whether part (i) can be proved in ZFC.

5 An almost continuous function $f : \mathbb{R} \to \mathbb{R}$

The aim of this section is to establish that an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ can exhibit complicated dynamical behavior stronger than topological transitivity even if all orbits are finite. First we obtain an auxiliary result.

Lemma 2. *Let X be an infinite, second countable metric space without isolated points, and let $a \in X$. Then there exists a function $f : X \to X$ such that*
(i) $f(a) = a$; and for every $x \in X$ there exists $n \in \mathbb{N}$ such that $f^n(x) = a$,
(ii) G_{f^m} is dense in X^2 for every $m \in \mathbb{N}$.

Proof. Let $\{B(j) : j \in \mathbb{N}\}$ be a countable base of nonempty open sets for X. Note that each $B(j)$ is infinite as X has no isolated points. Let $\{(i_k, j_k) : k \in \mathbb{N}\}$ be an enumeration of $\mathbb{N}^2$. We define $f : X \to X$ in an inductive fashion. Define $f(a) = a$ and put $D_0 = \{a\}$. Next, choose two distinct points $x_{1,0}, x_{1,1}$ in $X \setminus D_0$ such that $x_{1,0} \in B(i_1)$ and $x_{1,1} \in B(j_1)$. Define $f(x_{1,0}) = x_{1,1}$, $f(x_{1,1}) = a$ and put $D_1 = D_0 \cup \{x_{1,0}, x_{1,1}\}$. At the k^{th} step, choose $k + 1$ distinct points $x_{k,0}, x_{k,1}, \ldots, x_{k,k}$ in $X \setminus D_{k-1}$ such that $x_{k,0} \in B(i_k)$ and $x_{k,r} \in B(j_k)$ for $1 \le r \le k$. Define $f(x_{k,r}) = x_{k,r+1}$ for $0 \le r < k$, $f(x_{k,k}) = a$ and put $D_k = D_{k-1} \cup \{x_{k,r} : 0 \le r \le k\}$. Having defined f on $\bigcup_{k=0}^{\infty} D_k$, put $f(x) = a$ for $x \in X \setminus \bigcup_{k=0}^{\infty} D_k$. Clearly, statement (i) of the Lemma holds. Also, by construction we have that for any $k \in \mathbb{N}$, $f^r(B(i_k)) \cap B(j_k) \neq \emptyset$ for $1 \le r \le k$. Now, if U, V are nonempty subsets of X and $m \in \mathbb{N}$, we can find $k \ge m$ such that $B(i_k) \subset U$ and $B(j_k) \subset V$. Then, $f^m(U) \cap V \neq \emptyset$. □

A sufficient condition for a function $f : \mathbb{R} \to \mathbb{R}$ to be almost continuous, observed by K.R.Kellum in [5], is the following.

Proposition 6. *[5] Let $\mathcal{F}$ be the collection of all closed subsets F of $\mathbb{R}^2$ such that $\pi(F)$ has cardinality $\mathfrak{c}$, where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the projection to the first coordinate. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose graph G_f intersects every $F \in \mathcal{F}$. Then f is almost continuous.*

This helps us to show that:

Proposition 7. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following:*

(i) f^k *is almost continuous for every* $k \in \mathbb{N}$.

(ii) $f(0) = 0$*; and for every* $x \in \mathbb{R}$ *there exists* $n \in \mathbb{N}$ *such that* $f^n(x) = 0$,

(iii) G_{f^m} *is dense in* $\mathbb{R}^2$ *for every* $m \in \mathbb{N}$.

Proof. Applying Lemma 2 with $X = \mathbb{Q}$ and $a = 0$, we get a function $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(0) = 0$, that for every $x \in \mathbb{Q}$ there is $n \in \mathbb{N}$ with $f^n(x) = 0$, and such that G_{f^m} is dense in $\mathbb{Q}^2$ for every $m \in \mathbb{N}$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, it is clear that *any* extension of f to a function (for which we use the same notation) $f : \mathbb{R} \to \mathbb{R}$ will satisfy the statement (iii) of the Proposition. Hence it suffices to take care of statements (i) and (ii). We use transfinite induction to extend f from $\mathbb{Q}$ to $\mathbb{R}$.

Let $\mathcal{F}$ be the special collection of closed subsets of $\mathbb{R}^2$ mentioned in Proposition 6. We write $\mathcal{F} \times \mathbb{N} = \{(F_\alpha, k_\alpha) : \alpha < \mathfrak{c}\}$. To start the induction procedure, let $D_0 = \mathbb{Q}$.

Suppose that for some $\alpha < \mathfrak{c}$ and for all $\beta < \alpha$ we have chosen $D_\beta \subset \mathbb{R}$ such that D_β has cardinality less than $\mathfrak{c}$ and that f is defined on D_β. At the α^{th} step we do the following.

Consider (F_α, k_α). Since $\pi(F_\alpha)$ has cardinality $\mathfrak{c}$ and since $\bigcup_{\beta<\alpha} D_\beta$ has cardinality less than $\mathfrak{c}$, we may choose $k_\alpha + 1$ distinct points $\{x_{\alpha,j} : 0 \leq j \leq k_\alpha\}$ such that $x_{\alpha,j} \notin \bigcup_{\beta<\alpha} D_\beta$ for $0 \leq j < k_\alpha$ and such that $(x_{\alpha,0}, x_{\alpha,k_\alpha}) \in F_\alpha$. Define $f(x_{\alpha,j}) = x_{\alpha,j+1}$ for $0 \leq j < k_\alpha$. This ensures that $G_{f^j} \cap F_\alpha \neq \emptyset$ for $1 \leq j \leq k_\alpha$ so that Proposition 6 may be invoked later. If x_{α,k_α} does not belong to $\bigcup_{\beta<\alpha} D_\beta$, also define $f(x_{\alpha,k_\alpha}) = 0$. Put $D_\alpha = [\bigcup_{\beta<\alpha} D_\beta] \cup \{x_{\alpha,j} : 0 \leq j \leq k_\alpha\}$, which is again a set of cardinality less than $\mathfrak{c}$, thereby allowing the passage to the next step.

Finally, we define $f(x) = 0$ for $x \in \mathbb{R} \setminus \bigcup_{\alpha<\mathfrak{c}} D_\alpha$. That the resulting f works for us is easy verification. □

With slight modifications of the above proof one can get variants of Proposition 7. For instance, we can have:

Proposition 8. *There exists a function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the following:*

(i) f^k *is almost continuous for every* $k \in \mathbb{N}$.

(ii) the set $Fix(f)$ *of fixed points of* f *is dense in* $\mathbb{R}$,

(iii) for every $x \in \mathbb{R}$ *there exists* $n \in \mathbb{N}$ *such that* $f^n(x) \in Fix(f)$ *(thus every* f*-orbit is finite),*

(iv) G_{f^m} *is dense in* $\mathbb{R}^2$ *for every* $m \in \mathbb{N}$.

Hint for proof: Using Lemma 2, first obtain a function $f : \mathbb{Q} \to \mathbb{Q}$. Next, define $f(x) = x$ for $x \in \mathbb{Q} + \sqrt{2}$. Put $D_0 = \mathbb{Q} \cup [\mathbb{Q} + \sqrt{2}]$ and proceed by transfinite induction as in the proof of the previous Proposition. □

Question: Can the functions in the last two Propositions be extendable?

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